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# Rudin's submodules of $H^2(\mathbb{D}^2)$

*Sous-modules de Rudin de $H^2(\mathbb{D}^2)$*

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ABSTRACT

Let $\{\alpha_n\}_{n\geq 0}$ be a sequence of scalars in the open unit disc of $\mathbb{C}$, and let $\{l_n\}_{n\geq 0}$ be a sequence of natural numbers satisfying $\sum_{n=0}^{\infty}(1-l_n|\alpha_n|)<\infty$. Then the joint $(M_{z_1}, M_{z_2})$ invariant subspace

$$\mathcal{S}_\Phi = \bigvee_{n=0}^{\infty}\left(z_1^n \prod_{k=n}^{\infty}\left(\frac{-\bar{\alpha}_k}{|\alpha_k|}\,\frac{z_2-\alpha_k}{1-\bar{\alpha}_k z_2}\right)^{l_k} H^2(\mathbb{D}^2)\right),$$

is called a Rudin submodule. In this paper, we analyze the class of Rudin submodules and prove that

$$\dim\big(\mathcal{S}_\Phi \ominus (z_1\mathcal{S}_\Phi + z_2\mathcal{S}_\Phi)\big) = 1 + \#\{n\geq 0 : \alpha_n = 0\} < \infty.$$

In particular, this answers a question earlier raised by Douglas and Yang (2000) [4].



RÉSUMÉ

Soit $\{\alpha_n\}_{n\geq 0}$ une suite de scalaires du disque unité ouvert de $\mathbb{C}$, et soit $\{l_n\}_{n\geq 0}$ une suite de nombres naturels vérifiant $\sum_{n=0}^{\infty}(1-l_n|\alpha_n|)<\infty$. Alors le sous-espace invariant $(M_{z_1}, M_{z_2})$

$$S_\Phi = \bigvee_{n=0}^{\infty}\left(z_1^n \prod_{k=n}^{\infty}\left(\frac{-\bar{\alpha}_k}{|\alpha_k|}\,\frac{z_2-\alpha_k}{1-\bar{\alpha}_k z_2}\right)^{l_k} H^2(\mathbb{D}^2)\right),$$

est appelé sous-module de Rudin. Dans cette Note, on analyse la classe des sous-modules de Rudin et on démontre que

$$\dim\big(S_\Phi \ominus (z_1 S_\Phi + z_2 S_\Phi)\big) = 1 + \#\{n\geq 0 : \alpha_n = 0\} < \infty.$$

En particulier, ce résultat répond à une question posée précédemment par Douglas et Yang (2000) [4].



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## 1. Introduction

Let $H^2(\mathbb{D})$ denote the Hardy space over the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We also say that $H^2(\mathbb{D})$ is the *Hardy module* over $\mathbb{D}$. The Hilbert space tensor product $H^2(\mathbb{D}) \otimes H^2(\mathbb{D})$ is called the *Hardy module* over $\mathbb{D}^2$ and is denoted by $H^2(\mathbb{D}^2)$. As is well known, every vector in $H^2(\mathbb{D}^2)$ can be represented as square summable power series over $\mathbb{D}^2$ and the multiplication operators by the coordinate functions $(M_{z_1}, M_{z_2})$ are commuting and doubly commuting isometries (see [7]). We will often identify $(M_{z_1}, M_{z_2})$ with $(M_z \otimes I_{H^2(\mathbb{D})}, I_{H^2(\mathbb{D})} \otimes M_z)$.

A closed subspace $\mathcal{S}$ of $H^2(\mathbb{D}^2)$ (or $H^2(\mathbb{D})$) is said to be a *submodule* if $\mathcal{S}$ is invariant under $M_{z_1}$ and $M_{z_2}$ (or $M_z$). Beurling's (cf. [3]) celebrated result states that a closed subspace $\mathcal{S} \subseteq H^2(\mathbb{D})$ is a submodule of $H^2(\mathbb{D})$ if and only if $\mathcal{S} = \theta H^2(\mathbb{D})$ for some bounded inner function $\theta \in H^\infty(\mathbb{D})$. This is a fundamental result that has far-reaching consequences. For instance, it readily follows that $\mathcal{S} = \theta H^2(\mathbb{D})$ admits the wandering subspace $\mathcal{S} \ominus z\mathcal{S} = \mathbb{C}\theta$. In particular, $\mathcal{S} \ominus z\mathcal{S}$ is a generating set of $\mathcal{S}$. The same conclusion also holds when $\mathcal{S}$ is one of the following: a submodule of the Bergman module [1], a doubly commuting submodule of $H^2(\mathbb{D}^n)$ [8] and a doubly commuting submodule of the Bergman space or the Dirichlet space over polydisc [2]. This motivates us to look into the "wandering subspace" $\mathcal{W}_{\mathcal{S}} := \mathcal{S} \ominus (z_1\mathcal{S} + z_2\mathcal{S}) = (\mathcal{S} \ominus z_1\mathcal{S}) \cap (\mathcal{S} \ominus z_2\mathcal{S})$, and leads us to ask whether $\mathcal{W}_{\mathcal{S}}$ is a generating set of $\mathcal{S}$ or not, where $\mathcal{S}$ is a submodule of $H^2(\mathbb{D}^2)$. In general, however, this question has a negative answer. Rudin [7] demonstrated a negative answer to this question by constructing a submodule $\mathcal{S}$ of $H^2(\mathbb{D}^2)$ for which $\mathcal{S} \ominus (z_1\mathcal{S} + z_2\mathcal{S})$ is a finite-dimensional subspace, but not a generating set of $\mathcal{S}$.

Our motivation in this paper is the following: (1) to study a natural class of submodules, namely "generalized Rudin submodules", and (2) to compute the wandering dimensions of generalized Rudin submodules. In particular, we are interested in understanding the wandering subspace of Rudin submodules of $H^2(\mathbb{D}^2)$. Our results, restricted to the case of Rudin's submodule, answers a question raised by Douglas and Yang (see Corollary 3.2). Also these results are one important step in our program to understand the idea of constructing new submodules and quotient modules out of old ones.

Given an inner function $\varphi \in H^\infty(\mathbb{D})$, for notational simplicity, we set

$$\mathcal{S}_\varphi := \varphi H^2(\mathbb{D}), \quad \text{and} \quad \mathcal{Q}_\varphi := H^2(\mathbb{D}) \ominus \mathcal{S}_\varphi.$$

We now turn to formulate our definition of generalized Rudin submodules. Let $\Psi = \{\psi_n\}_{n=0}^\infty \subseteq H^\infty(\mathbb{D})$ be a sequence of increasing inner functions and $\Phi = \{\varphi_n\}_{n=0}^\infty \subseteq H^\infty(\mathbb{D})$ be a sequence of decreasing inner functions. Then the *generalized Rudin submodule* corresponding to the inner sequence $\Psi$ and $\Phi$ is denoted by $\mathcal{S}_{\Psi,\Phi}$, and defined by

$$\mathcal{S}_{\Psi,\Phi} = \bigvee_{n=0}^{\infty} (\mathcal{S}_{\psi_n} \otimes \mathcal{S}_{\varphi_n}).$$

Now let $\{\alpha_n\}_{n\geq 0}$ be a sequence of points in $\mathbb{D}$ and $\{l_i\}_{n=0}^\infty \subseteq \mathbb{N}$ such that $\sum(1 - l_i|\alpha_i|) < \infty$, and $\psi_n = z^n$, and $\varphi_n := \prod_{i=n}^\infty b_{\alpha_i}^{l_i}$, $n \geq 0$. Then $\mathcal{S}_{\Psi,\Phi}$ will be denoted by $\mathcal{S}_\Phi$:

$$\mathcal{S}_\Phi = \bigvee_{n=0}^{\infty} (\mathcal{S}_{z^n} \otimes \mathcal{S}_{\varphi_n}).$$

Here for each non-zero $\alpha \in \mathbb{D}$, we denote by $b_\alpha$ the Blaschke factor $b_\alpha(z) := \frac{-\bar{\alpha}}{|\alpha|} \frac{z-\alpha}{1-\bar{\alpha}z}$ and for $\alpha = 0$ we set $b_0(z) := z$.

The sequence of Blaschke products as defined above is called the *Rudin sequence*, and the submodule $\mathcal{S}_\Phi$ as defined above is called the *Rudin submodule* corresponding to the Rudin sequence $\Phi$. These submodules are also called inner-sequence-based invariant subspaces of $H^2(\mathbb{D}^2)$, and were studied by M. Seto and R. Yang [12], Seto [9–11], and Izuchi et al. [5].

The main result of this paper states that

$$\dim\big(\mathcal{S}_\Phi \ominus (z_1\mathcal{S}_\Phi + z_2\mathcal{S}_\Phi)\big) = 1 + \#\{n \geq 0 : \alpha_n = 0\} < \infty.$$

The remainder of the paper is organized as follows. Section 2 collects necessary notations and contains preparatory materials, which are an essential tool in what follows. After this preparatory section, which contains also new results, the main theorems are proved in Section 3.

## 2. Preparatory results

We begin with the following representations of $\mathcal{S}_{\Psi,\Phi}$ (cf. [5]).

**Lemma 2.1.** *Let $\mathcal{S}_{\Psi,\Phi}$ be a generalized Rudin submodule and $\varphi_{-1} := 0$. Then*

$$\mathcal{S}_{\Psi,\Phi} = \bigvee_{n=0}^{\infty} \mathcal{S}_{\psi_n} \otimes \mathcal{S}_{\varphi_n} = \bigoplus_{n=0}^{\infty} \mathcal{S}_{\psi_n} \otimes (\mathcal{S}_{\varphi_n} \ominus \mathcal{S}_{\varphi_{n-1}}). \tag{1}$$

**Proof.** First note that for all $n \geq 1$,

$$\bigoplus_{j=0}^{n} (\mathcal{S}_{\varphi_j} \ominus \mathcal{S}_{\varphi_{j-1}}) = \mathcal{S}_{\varphi_n}.$$

Then the required representation of $\mathcal{S}_{\Psi,\Phi}$ can be obtained from the above identity and the fact that $\mathcal{S}_{\psi_n} \subseteq \mathcal{S}_{\psi_{n-1}}$ $(n \geq 1)$.

Keeping the equality $\mathcal{S}_{\Psi,\Phi} \ominus (z_1 \mathcal{S}_{\Psi,\Phi} + z_2 \mathcal{S}_{\Psi,\Phi}) = (\mathcal{S}_{\Psi,\Phi} \ominus z_1 \mathcal{S}_{\Psi,\Phi}) \cap (\mathcal{S}_{\Psi,\Phi} \ominus z_2 \mathcal{S}_{\Psi,\Phi})$ in mind, we pass to describe the closed subspace $(\mathcal{S}_{\Psi,\Phi} \ominus z_1 \mathcal{S}_{\Psi,\Phi})$.

**Lemma 2.2.** *Let $\mathcal{S}_{\Psi,\Phi}$ be a generalized Rudin's submodule and $\varphi_{-1} := 0$. Then*

$$(\mathcal{S}_{\Psi,\Phi} \ominus z_1 \mathcal{S}_{\Psi,\Phi}) = \bigoplus_{n=0}^{\infty} \mathbb{C}\psi_n \otimes (\mathcal{S}_{\varphi_n} \ominus \mathcal{S}_{\varphi_{n-1}}).$$

**Proof.** By Lemma 2.1 it follows that

$$\begin{aligned}
\mathcal{S}_{\Psi,\Phi} \ominus z_1 \mathcal{S}_{\Psi,\Phi} &= \left( \bigoplus_{n=0}^{\infty} \mathcal{S}_{\psi_n} \otimes (\mathcal{S}_{\varphi_n} \ominus \mathcal{S}_{\varphi_{n-1}}) \right) \ominus z_1 \left( \bigoplus_{n=0}^{\infty} \mathcal{S}_{\psi_n} \otimes (\mathcal{S}_{\varphi_n} \ominus \mathcal{S}_{\varphi_{n-1}}) \right) \\
&= \bigoplus_{n=0}^{\infty} (\mathcal{S}_{\psi_n} \ominus z_1 \mathcal{S}_{\psi_n}) \otimes (\mathcal{S}_{\varphi_n} \ominus \mathcal{S}_{\varphi_{n-1}}).
\end{aligned}$$

Thus the result follows from the fact that $\mathcal{S}_\theta \ominus z\mathcal{S}_\theta = \mathbb{C}\theta$, for any inner $\theta \in H^\infty(\mathbb{D})$.

Before proceeding further, we first observe that for $\alpha \in \mathbb{D}$ and $m \geq 1$, $\{b_\alpha^j M_z^* b_\alpha\}_{j=0}^{m-1}$ is an orthogonal basis of the quotient module $\mathcal{Q}_{b_\alpha^m}$.

**Lemma 2.3.** *Let $\theta_1, \theta_2$ be a pair of inner functions such that $\theta_1 = b_\alpha^m \theta_2$ for some $\alpha \in \mathbb{D}$ and $m \geq 1$. Then*

$$\mathcal{S}_{\theta_2} \ominus \mathcal{S}_{\theta_1} = \theta_2 \mathcal{Q}_{b_\alpha^m} = \theta_2 \big( H^2(\mathbb{D}) \ominus b_\alpha^m H^2(\mathbb{D}) \big) = \bigoplus_{k=0}^{m-1} \mathbb{C}\theta_2 \big( b_\alpha^k M_z^* b_\alpha \big).$$

*In particular, $\mathcal{S}_{\theta_2} \ominus \mathcal{S}_{\theta_1}$ is an $m$-dimensional subspace of $H^2(\mathbb{D})$.*

**Proof.** The proof follows from the fact that $M_{\theta_2}$ is an isometry and $\{b_\alpha^j M_z^* b_\alpha\}_{j=0}^{m-1}$ is an orthogonal basis of $\mathcal{Q}_{b_\alpha^m}$.

To proceed with our discussion, it is useful to compute the matrix representation of the operator $P_{\mathcal{S}_{\theta_2} \ominus \mathcal{S}_{\theta_1}} M_z^*|_{\mathcal{S}_{\theta_2} \ominus \mathcal{S}_{\theta_1}}$ with respect to the orthogonal basis $\{\theta_2 b_\alpha^j M_z^* b_\alpha\}_{j=0}^{m-1}$, which will be used in the proof of the main result of this paper. To this end, let $v_j = \theta_2 b_\alpha^j M_z^* b_\alpha$ for all $j = 0, \ldots, m-1$. Then

$$\big\langle \big( P_{\mathcal{S}_{\theta_2} \ominus \mathcal{S}_{\theta_1}} M_z^* \big) v_j, v_i \big\rangle = \big\langle M_z^* b_\alpha^j M_z^* b_\alpha, b_\alpha^i M_z^* b_\alpha \big\rangle.$$

Note that

$$M_z^* b_\alpha = -\frac{\bar{\alpha}}{|\alpha|} \big( 1 - |\alpha|^2 \big) \mathbb{S}(\cdot, \alpha),$$

where $\mathbb{S}(\cdot, \alpha)$ is the Szegö kernel on $\mathbb{D}$ defined by $\mathbb{S}(\cdot, \alpha)(z) = (1 - \bar{\alpha} z)^{-1}$, $z \in \mathbb{D}$. Consequently, for $i = j$, we have:

$$\big\langle \big( P_{\mathcal{S}_{\theta_2} \ominus \mathcal{S}_{\theta_1}} M_z^* \big) v_j, v_i \big\rangle = \big\langle M_z^{*2} b_\alpha, M_z^* b_\alpha \big\rangle = \bar{\alpha} \big( 1 - |\alpha|^2 \big),$$

for $i > j$,

$$\big\langle \big( P_{\mathcal{S}_{\theta_2} \ominus \mathcal{S}_{\theta_1}} M_z^* \big) v_j, v_i \big\rangle = \big\langle M_z^* b_\alpha^j M_z^* b_\alpha, b_\alpha^i M_z^* b_\alpha \big\rangle = \big\langle M_z^{*2} b_\alpha, b_\alpha^{i-j} M_z^* b_\alpha \big\rangle = 0,$$

and for $j > i$,

$$\begin{aligned}
\big\langle \big( P_{\mathcal{S}_{\theta_2} \ominus \mathcal{S}_{\theta_1}} M_z^* \big) v_j, v_i \big\rangle &= \big\langle M_z^* b_\alpha^j M_z^* b_\alpha, b_\alpha^i M_z^* b_\alpha \big\rangle = \big\langle M_z^* b_\alpha^{j-i} M_z^* b_\alpha, M_z^* b_\alpha \big\rangle \\
&= \big( 1 - |\alpha|^2 \big)^2 \big\langle M_z^* b_\alpha^{j-i} \mathbb{S}(\cdot, \alpha), \mathbb{S}(\cdot, \alpha) \big\rangle = (-\alpha)^{j-i-1} \big( 1 - |\alpha|^2 \big)^2,
\end{aligned}$$

where the last equality follows from

$$\begin{aligned}\langle b_\alpha^{j-i}\mathbb{S}(\cdot,\alpha), z\mathbb{S}(\cdot,\alpha)\rangle &= \langle b_\alpha^{j-i}\mathbb{S}(\cdot,\alpha), b_\alpha+\alpha\mathbb{S}(\cdot,\alpha)\rangle = \langle b_\alpha^{j-i}\mathbb{S}(\cdot,\alpha), b_\alpha\rangle + \bar{\alpha}\langle b_\alpha^{j-i}\mathbb{S}(\cdot,\alpha),\mathbb{S}(\cdot,\alpha)\rangle \\ &= \langle b_\alpha^{j-i-1}\mathbb{S}(\cdot,\alpha),\mathbb{S}(\cdot,0)\rangle + \bar{\alpha}\big(b_\alpha^{j-i}\mathbb{S}(\cdot,\alpha)\big)(\alpha) = \big(b_\alpha^{j-i-1}\mathbb{S}(\cdot,\alpha)\big)(0)+0 \\ &= (-\alpha)^{j-i-1}.\end{aligned}$$

Therefore,

$$\big\langle \big(P_{\mathcal{S}_{\theta_2}\ominus\mathcal{S}_{\theta_1}} M_z^*|_{\mathcal{S}_{\theta_2}\ominus\mathcal{S}_{\theta_1}}\big)v_j, v_i\big\rangle = \begin{cases} 0 & \text{if } j<i, \\ \bar{\alpha}(1-|\alpha|^2) & \text{if } j=i, \\ (-\alpha)^{j-i-1}(1-|\alpha|^2)^2 & \text{if } j\ge i+1. \end{cases}$$

Finally, since $M_{\theta_2 b_\alpha^i}\in\mathcal{B}(H^2(\mathbb{D}))$ is an isometry for any $0\le i\le m-1$, we have:

$$\|v_i\| = \big\|\theta_2 b_\alpha^i M_z^* b_\alpha\big\| = \big\|M_z^* b_\alpha\big\| = \big(1-|\alpha|^2\big)\big\|\mathbb{S}(\cdot,\alpha)\big\| = \big(1-|\alpha|^2\big)^{\frac{1}{2}}.$$

The computations above then show that the matrix representation of $P_{\mathcal{S}_{\theta_2}\ominus\mathcal{S}_{\theta_1}} M_z^*|_{\mathcal{S}_{\theta_2}\ominus\mathcal{S}_{\theta_1}}$ with respect to the orthonormal basis $\{\frac{1}{\sqrt{1-|\alpha|^2}}v_j\}_{j=0}^{m-1}$ is the upper triangular matrix with diagonal entries $\bar{\alpha}$ and off diagonal entries $(-\alpha)^{j-i-1}(1-|\alpha|^2)$.

## 3. Main results

**Theorem 3.1.** *Let $\mathcal{S}_\Phi$ be a Rudin submodule of $H^2(\mathbb{D}^2)$. Then*

$$\dim\big(\mathcal{S}_\Phi\ominus(z_1\mathcal{S}_\Phi+z_2\mathcal{S}_\Phi)\big) = 1+\#\{n\ge 0:\alpha_n=0\}<\infty.$$

**Proof.** Since $\{n\ge 0:\alpha_n=0\}$ is a finite set, it is enough to show that the equality holds. First, observe that

$$\mathcal{S}_\Phi\ominus(z_1\mathcal{S}_\Phi+z_2\mathcal{S}_\Phi) = (\mathcal{S}_\Phi\ominus z_1\mathcal{S}_\Phi)\cap\big(\ker P_{\mathcal{S}_\Phi}M_{z_2}^*\big|_{\mathcal{S}_\Phi}\big) = \ker P_{\mathcal{S}_\Phi}M_{z_2}^*\big|_{\mathcal{S}_\Phi\ominus z_1\mathcal{S}_\Phi}.$$

Now Lemmas 2.1 and 2.2, with $\psi_n=z^n$, $n\ge 0$, and $\varphi_{-1}=0$ imply that

$$\mathcal{S}_\Phi = \bigoplus_{n\ge 0}\big(z^n H^2(\mathbb{D})\otimes(\mathcal{S}_{\varphi_n}\ominus\mathcal{S}_{\varphi_{n-1}})\big),$$

and

$$\mathcal{S}_\Phi\ominus z_1\mathcal{S}_\Phi = \bigoplus_{n\ge 0}\big(\mathbb{C}z^n\otimes(\mathcal{S}_{\varphi_n}\ominus\mathcal{S}_{\varphi_{n-1}})\big).$$

Then

$$\begin{aligned}P_{\mathcal{S}_\Phi}M_{z_2}^*(\mathcal{S}_\Phi\ominus z_1\mathcal{S}_\Phi) &= P_{\mathcal{S}_\Phi}\Big(\bigoplus_{n\ge 0}\big(\mathbb{C}z^n\otimes M_z^*(\mathcal{S}_{\varphi_n}\ominus\mathcal{S}_{\varphi_{n-1}})\big)\Big) \\ &= \bigoplus_{n\ge 0}\big(\mathbb{C}z^n\otimes P_{\mathcal{S}_{\varphi_n}\ominus\mathcal{S}_{\varphi_{n-1}}}M_z^*(\mathcal{S}_{\varphi_n}\ominus\mathcal{S}_{\varphi_{n-1}})\big),\end{aligned}$$

where for the last equality we have used the fact that $P_{\mathcal{S}_{\varphi_{n-1}}}M_z^*(\mathcal{S}_{\varphi_n}\ominus\mathcal{S}_{\varphi_{n-1}})=\{0\}$. Therefore,

$$\ker P_{\mathcal{S}_\Phi}M_{z_2}^*\big|_{\mathcal{S}_\Phi\ominus z_1\mathcal{S}_\Phi} = \big(\mathbb{C}\otimes\ker P_{\mathcal{S}_{\varphi_0}}M_z^*\big|_{\mathcal{S}_{\varphi_0}}\big)\bigoplus_{n=1}^{\infty}\big(\mathbb{C}z^n\otimes\ker\big(P_{\mathcal{S}_{\varphi_n}\ominus\mathcal{S}_{\varphi_{n-1}}}M_z^*\big|_{\mathcal{S}_{\varphi_n}\ominus\mathcal{S}_{\varphi_{n-1}}}\big)\big).$$

For the first term on the right-hand side, we have $\ker P_{\mathcal{S}_{\varphi_0}}M_z^*|_{\mathcal{S}_{\varphi_0}}=\mathbb{C}\varphi_0$, that is,

$$\dim\big(\ker P_{\mathcal{S}_{\varphi_0}}M_z^*\big|_{\mathcal{S}_{\varphi_0}}\big)=1.$$

On the other hand, the representing matrix of $P_{\mathcal{S}_{\varphi_n}\ominus\mathcal{S}_{\varphi_{n-1}}}M_z^*|_{\mathcal{S}_{\varphi_n}\ominus\mathcal{S}_{\varphi_{n-1}}}$ with respect to the orthonormal basis $\{\frac{1}{\sqrt{1-|\alpha_{n-1}|^2}}\cdot\varphi_n b_{\alpha_{n-1}}^k M_z^* b_{\alpha_{n-1}}:0\le k<l_n\}$, as discussed at the end of the previous section, is an upper triangular matrix with $\bar{\alpha}_{n-1}$ on the diagonal. This matrix is invertible if and only if $\alpha_{n-1}\ne 0$. In the case of $\alpha_{n-1}=0$, since the supper diagonal entries are 1, the rank of the representing matrix is $l_n-1$ and hence the kernel is one-dimensional. This completes the proof. □

As an illustration of the above theorem, we consider an explicit example of Rudin submodule. Let $\mathcal{S}_R$ denote the submodule of $H^2(\mathbb{D}^2)$, consisting of those functions that have zero of order at least $n$ at $(0, \alpha_n) := (0, 1 - n^{-3})$. This submodule was introduced by W. Rudin in the context of infinite rank submodules of $H^2(\mathbb{D}^2)$. It is also well known that such $\mathcal{S}_R$ is a Rudin submodule (see [9,10,12]), that is, $\mathcal{S}_R = \mathcal{S}_\Phi$ where

$$\varphi_0 = \prod_{i=1}^{\infty} b^i_{\alpha_i}, \qquad \varphi_n = \frac{\varphi_{n-1}}{\prod_{j=n}^{\infty} b_{\alpha_j}} \quad (n \geq 1).$$

In this case, for all $n \geq 1$, $\mathcal{S}_{\varphi_n} \ominus \mathcal{S}_{\varphi_{n-1}} = \varphi_n(H^2(\mathbb{D}) \ominus \prod_{j=n}^{\infty} b_{\alpha_j} H^2(\mathbb{D}))$. Set

$$e_k := \left( \prod_{j=k+1}^{\infty} b_{\alpha_j} \right) M_z^* b_{\alpha_k} \quad (k \geq 1).$$

Then $\{\varphi_n e_k\}_{k=n}^{\infty}$ is an orthogonal basis for $\mathcal{S}_{\varphi_n} \ominus \mathcal{S}_{\varphi_{n-1}}$, for all $n \geq 1$. A similar calculation, as in the end of the previous section, shows that the matrix representation of the operator $P_{\mathcal{S}_{\varphi_n} \ominus \mathcal{S}_{\varphi_{n-1}}} M_z^*|_{\mathcal{S}_{\varphi_n} \ominus \mathcal{S}_{\varphi_{n-1}}}$ with respect to $\{\varphi_n e_k\}_{k=n}^{\infty}$ is again an upper triangular infinite matrix with diagonal entries $\bar{\alpha}_k$ $(k \geq n)$. Therefore, $\ker P_{\mathcal{S}_{\varphi_n} \ominus \mathcal{S}_{\varphi_{n-1}}} M_z^*|_{\mathcal{S}_{\varphi_n} \ominus \mathcal{S}_{\varphi_{n-1}}} = \{0\}$ for all $n > 1$. Now for $n = 1$, since $\alpha_1 = 0$ its kernel is one-dimensional. By the same argument, as in the proof of the above theorem, we obtain the following result.

**Corollary 3.2.** *Let $\mathcal{S}_R$ be the Rudin submodule as above. Then $\mathcal{S}_R \ominus (z_1\mathcal{S}_R + z_2\mathcal{S}_R)$ is a two-dimensional subspace given by*

$$\mathcal{S}_R \ominus (z_1\mathcal{S}_R + z_2\mathcal{S}_R) = \mathbb{C} \prod_{i=1}^{\infty} b^i_{\alpha_i} \oplus \mathbb{C} \prod_{i=2}^{\infty} b^i_{\alpha_i}.$$

We end this note with an intriguing question, raised by Nakazi [6]: Does there exist a submodule $\mathcal{S}$ of $H^2(\mathbb{D}^2)$ with rank $\mathcal{S} = 1$, such that $\mathcal{S} \ominus (z_1\mathcal{S} + z_2\mathcal{S})$ is not a generating set for $\mathcal{S}$? Although we are unable to determine such (counter-)example, however, we do have the following special example: let $\{\alpha_n\}_{n=0}^{\infty} \subseteq \mathbb{D} \setminus \{0\}$ be a sequence of distinct points and $\Phi = \{\varphi_n\}_{n \geq 0}$, and $\varphi_n := \prod_{i=n}^{\infty} b^{l_i}_{\alpha_i}$. Then, by the main result of this paper, $\mathcal{S}_\Phi \ominus (z_1\mathcal{S}_\Phi + z_2\mathcal{S}_\Phi) = \mathbb{C} \otimes \mathbb{C}\varphi_0$ is a one-dimensional, non-generating subspace of $\mathcal{S}_\Phi$. In fact, it follows from [5] that the rank of $\mathcal{S}_\Phi$ is 2.

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