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Multiple zeta values at the non-positive integers

Les valeurs de la fonction zêta multiple aux entiers négatifs

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ABSTRACT

In this paper, we provide an alternative method to calculate the multiple zeta values at non-positive integers by means of Raabe's formula and the Bernoulli numbers.

RÉSUMÉ

Dans cet article, nous proposons une autre méthode pour calculer les valeurs de la fonction zêta multiple aux entiers négatifs à l'aide de la formule de Raabe et des nombres de Bernoulli.

0. Introduction and notations

The multiple Zeta values due to D. Zagier are defined by $\zeta_k(s_1,\dots,s_k)=\sum_{0<n_1<\cdots<n_k}\frac{1}{n_1^{s_1}\dots n_n^{s_n}}$ for positive integers s_i $(i=1,\dots,k)$ and $s_k\geq 2$. These values have a certain connection with topology and physics, and algebraic relations among them are extensively studied (see [14,15] and [4]). Euler evaluated several special cases of double zeta values. Further, he showed a very interesting formula which is at the origin of the sum formula in [5]. At present, a number of relations among multiple zeta values is known, for instance the sum formula [5,8], Hoffman's relation [10], and Ohno's relation [13]. On the other hand, Atkinson [3] investigated the analytic properties of the double zeta function and gave the analytic continuation of $\zeta_2(s_1,s_2)$ to consider the mean square of the Riemann zeta function on the critical line. Atkinson's essential tool is the Euler–Maclaurin summation formula and the Poisson summation formula. The general meromorphic continuation of $\zeta_k(s_1,\dots,s_k)$ was given by Zhao [16] using distribution theory and independently by Akiyama, Egami and Tanigawa [1] by means of the Euler–Maclaurin summation formula. Further, Matsumoto [12] showed the meromorphic continuation of a more general multiple zeta function by means of the Mellin–Barnes integral.

In this paper, we are interested in similar expressions for the multiple zeta values at non-positive integers. This issue has been discussed by various authors (see [11,9], and [1]) who all use the Euler–Maclaurin formula to deduce multiple values at non-positive integers from the corresponding iterated integrals. The Euler–Maclaurin formula expresses the discrete sum in terms of the integral and an interpolating function. Here, instead we use the Raabe formula [7], which expresses the integral in terms of the sum.

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In what follows, for any elements $\underline{\mathbf{x}} = (x_1, \ldots, x_n)$ and $\underline{\mathbf{y}} = (y_1, \ldots, y_n)$ of $\mathbb{C}^n$, we write $\|\underline{\mathbf{x}}\| = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$, $|\underline{\mathbf{x}}| = |x_1| + \cdots + |x_n|$ and $\langle \underline{\mathbf{x}}, \underline{\mathbf{y}} \rangle = x_1 y_1 + \cdots + x_n y_n$ the standard scalar product. We denote the canonical basis of $\mathbb{R}^n$ by $(\underline{\mathbf{e}}_1, \ldots, \underline{\mathbf{e}}_n)$, $\underline{\mathbf{s}} = (s_1, \ldots, s_r)$ denote a vector in $\mathbb{C}^r$, and we write $\underline{\mathbf{s}} = \underline{\boldsymbol{\sigma}} + i\underline{\boldsymbol{\tau}}$, where $\underline{\boldsymbol{\sigma}} = (\sigma_1, \ldots, \sigma_r)$ and $\underline{\boldsymbol{\tau}} = (\tau_1, \ldots, \tau_r)$ are the real (resp. imaginary) components of $\underline{\mathbf{s}}$ (i.e $\sigma_i = \Re(s_i)$ and $\tau_i = \Im(s_i)$ for all i).

The notation $f(\lambda, y, x) \ll_y g(x)$ (uniformly at $x \in X$ and $\lambda \in \Lambda$) stands for the existence of $A = A(y) > 0$, which depends neither on x nor λ, but could depend on the parameter vector y, such that:

$$\left| f(\lambda, y, x) \right| \le A.g(y) \quad \text{for any } x \in X \text{ and any } \lambda \in \Lambda. \tag{0.1}$$

1. Main results

$$\text{Let } \underline{s} = (s_1, \ldots, s_n) \in \mathbb{C}^n, \text{ we denote the multiple zeta function by } Z(\underline{s}) = \sum_{\underline{m} \in \mathbb{N}^{*n}} \prod_{i=1}^{n} \frac{1}{(m_1 + \cdots + m_i)^{s_i}} \tag{1.1}$$

$$\text{and the corresponding integral function by } Y(\underline{s}) = \int_{[1,+\infty[^n} \prod_{i=1}^{n} \frac{1}{(x_1 + \cdots + x_i)^{s_i}} \mathrm{d}\underline{x}. \tag{1.2}$$

For the meromorphic continuation of the integral (1.2) and the series (1.1), we refer the reader to the work of Akiyama [1].

We first give well-known elementary result for the integral function.

Lemma 1.1. *Let $\underline{N} = (N_1, \ldots, N_n)$ be a point of $\mathbb{N}^n$.*

(1) *The point $(\underline{s} = -\underline{N})$ is a polar divisor for the function $Y(\underline{s})$ if and only if there exists a $\underline{k} = (k_2, \ldots, k_n) \in \mathbb{N}^{n-1}$ such that*

$$(s_n - 1)(s_n + s_{n-1} - 2 + k_n) \ldots \left(\sum_{i=1}^{n} s_i - n + \sum_{i=2}^{n} k_i \right) = \prod_{j=1}^{n} \left(\sum_{i=j}^{n} s_i - n + j - 1 + \sum_{i=j+1}^{n} k_i \right) = 0. \tag{1.3}$$

(2) *If $(\underline{s} = -\underline{N})$ is not a polar divisor for the integral function, then the value of this function at this point exists and is given by*

$$Y(-\underline{N}) = \sum_{\underline{k} \in T(\underline{N})} \prod_{j=2}^{n} \frac{\binom{\sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i}{k_j} (-1)^n}{(\sum_{i=1}^{n} s_i - n + \sum_{i=2}^{n} k_i)(\sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i)} \tag{1.4}$$

with

$$T(\underline{N}) := \left\{ \underline{k} = (k_2, \ldots, k_n) \in \mathbb{N}^{n-1} : 0 \le k_j \le \sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i, \ \forall 2 \le j \le n \right\}. \tag{1.5}$$

We give now a similar result for the multiple zeta function.

Theorem 1. *Let $\underline{N} = (N_1, \ldots, N_n)$ a point of $\mathbb{N}^n$, if the point $(\underline{s} = -\underline{N})$ is not a polar divisor for the integral function $Y(\underline{s})$, then the value of the multiple zeta function $Z(\underline{s})$ at the point $(\underline{s} = -\underline{N})$ exists and is given by*

$$Z(-\underline{N}) = \sum_{\underline{k} = (k_2, \ldots, k_n) \in T(\underline{N})} \sum_{\substack{\underline{v} = (v_1, \ldots, v_n) \in \mathbb{N}^n \\ v_j \le k_j \ \forall 2 \le j \le n;\, v_1 \le (\sum_{i=1}^{n} N_i + n - \sum_{i=2}^{n} k_i)}} \prod_{j=2}^{n} \frac{(-1)^n A(-\underline{N}) B_{\underline{v}}}{(\sum_{i=1}^{n} N_i + n - \sum_{i=2}^{n} k_i)(\sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i)} \tag{1.6}$$

with

$$A(-\underline{N}) = \binom{\sum_{i=1}^{n} N_i + n - \sum_{i=2}^{n} k_i}{v_1} \prod_{j=2}^{n} \binom{\sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i}{k_j} \binom{k_j}{v_j}$$

and $T(\underline{N}) := \left\{ \underline{k} = (k_2, \ldots, k_n) \in \mathbb{N}^{n-1} : 0 \le k_j \le \sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i, \ \forall 2 \le j \le n \right\}$

and $B_{\underline{v}} = \prod_{j=1}^{n} B_{v_j}$ where B_{v_j} is the v_j-th Bernoulli number.

Proof of Theorem 1 in the case $n = 1$. In this part, we give the proof of our main result for $n = 1$ as a warm up for the proof for larger n. We have

$$\zeta(s) = \sum_{n>0} \frac{1}{n^s} = Z(s). \tag{1.7}$$

For $a \in \mathbb{R}_+$ we set $Y_a(s) = \int_1^{+\infty} (x+a)^{-s} \mathrm{d}x$ (1.8)

which for $\Re(s) > 1$ reads $Y_a(s) = \int_1^{+\infty} (x+a)^{-s} \mathrm{d}x = \frac{(1+a)^{-s+1}}{s-1}$. (1.9)

Thus, for all $N \in \mathbb{N}$: $Y_a(-N) = -\frac{(1+a)^{N+1}}{N+1} = \frac{-1}{N+1} \sum_{k=0}^{N+1} \binom{N+1}{k} a^k$. (1.10)

Then, Proposition 4.2 of Section 4 shows that $\zeta(-N) = Z(-N) = \frac{-1}{N+1} \sum_{k=0}^{N+1} \binom{N+1}{k} B_k$ (1.11)

where B_k is the k-th Bernoulli number, which ends the proof of Theorem 1 for $n = 1$.

Now, we recall the elementary result: $(N+1)B_N = -\sum_{k=0}^{N-1} \binom{N+1}{k} B_k$. (1.12)

Finally, we obtain the known result: $\zeta(-N) = Z(-N) = -\frac{B_{N+1}}{N+1}$. □ (1.13)

2. Proof of Lemma 1.1

Let the integral function: $Y(\underline{\mathbf{s}}) = \int_{[1,+\infty[^n} \prod_{i=1}^{n} (x_1 + \cdots + x_i)^{-s_i} \, \mathrm{d}\underline{x}$. (2.1)

We use the following change of variables: $y_i = x_1 + \cdots + x_i - (i-1)$ (2.2)

for all $1 \le i \le n$, which gives $x_1 = y_1$ and $x_i = y_i - y_{i-1} + 1$ (2.3)

for all $2 \le i \le n$.

Since $\underline{x} = (x_1, \ldots, x_n) \in [1, +\infty[^n$, this gives $\underline{y} \in V_n = \{\underline{y} \in \mathbb{R}^n : \ 1 \le y_1 \le y_2 \le \cdots \le y_n\}$ (2.4)

and, we find $Y(\underline{\mathbf{s}}) = \int_{V_n} \prod_{i=1}^{n} (y_i + i - 1)^{-s_i} \, \mathrm{d}\underline{y}$. (2.5)

This integral can be rewritten as follows: $Y(\underline{\mathbf{s}}) = \int_{V_{n-1}} \prod_{i=1}^{n-1} (y_i + i - 1)^{-s_i} \left(\int_{y_{n-1}}^{+\infty} (y_n + n - 1)^{-s_n} \, \mathrm{d}y_n \right) \mathrm{d}y_1 \ldots \mathrm{d}y_{n-1}$ (2.6)

with

$$\int_{y_{n-1}}^{+\infty} (y_n + n - 1)^{-s_n} \, \mathrm{d}y_n = \frac{(y_{n-1} + n - 2)^{-s_n+1}}{s_n - 1} \left(1 + \frac{1}{y_{n-1} + n - 2}\right)^{-s_n+1}$$

$$= \sum_{k_n \in \mathbb{N}} \binom{-s_n + 1}{k_n} \frac{(y_{n-1} + n - 2)^{-s_n+1-k_n}}{s_n - 1} \quad \text{if and only if} \quad \Re(s_n) - 1 > 0. \tag{2.7}$$

Inductively on n, we find:

$$Y(\underline{s}) = \sum_{\underline{k}=(k_2,\ldots,k_n) \in \mathbb{N}^{n-1}} \frac{\binom{-s_n+1}{k_n}\binom{-s_n-s_{n-1}+2-k_n}{k_{n-1}} \ldots \binom{-\sum_{i=2}^{n} s_i + n - \sum_{i=3}^{n} k_i}{k_2}}{(s_n - 1)(s_n + s_{n-1} - 2 + k_n) \ldots (\sum_{i=1}^{n} s_i - n + \sum_{i=2}^{n} k_i)} \tag{2.8}$$

if and only if for all $1 \le i \le n-1$, $\Re\left(\sum_{i=1}^{n} s_i\right) - n + j - 1 + \sum_{i=2}^{n} k_i > 0$ (2.9)

and $\Re(s_n) - 1 > 0$. (2.10)

Therefore, for any point $\underline{N} = (N_1, \ldots, N_n) \in \mathbb{N}^n$

1) the point $(\underline{s} = -\underline{N})$ is a polar divisor for the function $Y(\underline{s})$ if there exists a $\underline{k} = (k_2, ..., k_n) \in \mathbb{N}^{n-1}$ such that

$$(s_n - 1)(s_n + s_{n-1} - 2 + k_n)\ldots\left(\sum_{i=1}^{n} s_i - n + \sum_{i=2}^{n} k_i\right) = \prod_{j=1}^{n}\left(\sum_{i=j}^{n} s_i - n + j - 1 + \sum_{i=j+1}^{n} k_i\right) = 0; \tag{2.11}$$

2) if $(\underline{s} = -\underline{N})$ is not a polar divisor, we get

$$\binom{N_n + 1}{k_n}\ldots\binom{\sum_{i=2}^{n} N_i + n - \sum_{i=3}^{n} k_i}{k_2} = \prod_{j=2}^{n}\binom{\sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i}{k_j} = 0 \tag{2.12}$$

if and only if there exists a $\underline{k} = (k_2, ..., k_n) \in \mathbb{N}^{n-1}$ and $2 \le j \le n$, such that $k_j > \sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i$.
Let

$$T(\underline{N}) := \left\{\underline{k} = (k_2, ..., k_n) \in \mathbb{N}^{n-1} : \ 0 \le k_j \le \sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i, \ \forall 2 \le j \le n\right\} \tag{2.13}$$

which is finite, then

$$Y(-\underline{N}) = \sum_{\underline{k} \in T(\underline{N})} \prod_{j=2}^{n} \frac{\binom{\sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i}{k_j}(-1)^n}{(\sum_{i=1}^{n} s_i - n + \sum_{i=2}^{n} k_i)(\sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i)}. \tag{2.14}$$

3. An intermediate estimate

For $\underline{a} = (a_1, ..., a_n) \in \mathbb{R}_+^n$ and $\underline{s} = (s_1, ..., s_n) \in \mathbb{C}^n$, we define the function:

$$Y_{\underline{a}}(\underline{s}) = \int_{[1,+\infty[^n} \prod_{i=1}^{n} (x_1 + \cdots + x_i + a_1 + \cdots + a_i)^{-s_i} \, d\underline{x}. \tag{3.1}$$

We prove the following useful result.

Proposition 3.1. *Let $\underline{N} = (N_1, ..., N_n)$ a point of $\mathbb{N}^n$, then we have for $\underline{a} \in \mathbb{R}^+$:*

$$Y_{\underline{a}}(-\underline{N}) = \sum_{\underline{k}=(k_2,...,k_n) \in T(\underline{N})} \ \sum_{\substack{\underline{v}=(v_1,...,v_n) \in \mathbb{N}^n \\ v_j \le k_j \ \forall 2 \le j \le n;\, v_1 \le (\sum_{i=1}^{n} N_i + n - \sum_{i=2}^{n} k_i)}} \ \prod_{j=2}^{n} \frac{(-1)^n A(-\underline{N}) a_1^{v_1} a_j^{v_j}}{(\sum_{i=1}^{n} N_i + n - \sum_{i=2}^{n} k_i)(\sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i)} \tag{3.2}$$

with

$$A(-\underline{N}) = \binom{\sum_{i=1}^{n} N_i + n - \sum_{i=2}^{n} k_i}{v_1} \prod_{j=2}^{n} \binom{\sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i}{k_j} \binom{k_j}{v_j} \tag{3.3}$$

and

$$T(\underline{N}) := \left\{\underline{k} = (k_2, ..., k_n) \in \mathbb{N}^{n-1} : \ 0 \le k_j \le \sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i, \ \forall 2 \le j \le n\right\}. \tag{3.4}$$

Proof. Let $\underline{a} \in \mathbb{R}_+^n$, such that for all $\underline{x} = (x_1, ..., x_n) \in [1, +\infty[^n$ and for all $1 \le i \le n$

$$\frac{1 + a_i}{x_1 + ... + x_{i-1} + a_1 + ... + a_{i-1}} < 1, \tag{3.5}$$

we have: $Y_{\underline{a}}(\underline{s}) = \displaystyle\int_{[1,+\infty[^n} \prod_{i=1}^{n} (x_1 + \cdots + x_i + a_1 + \cdots + a_i)^{-s_i} \, d\underline{x}.$ (3.6)

This integral can be written as follows:

$$Y_{\underline{a}}(\underline{s}) = \int\limits_{[1,+\infty[^{n-1}} \prod_{i=1}^{n-1} (x_1 + \cdots + x_i + a_1 + \cdots + a_i)^{-s_i} \left(\int\limits_1^{+\infty} (x_1 + \cdots + x_n + a_1 + \cdots + a_n)^{-s_n} \, dx_n \right) dx_1 \ldots dx_{n-1}. \tag{3.7}$$

Since for $\Re(s_n) > 1$, we have:

$$\int\limits_1^{+\infty} (x_1 + \cdots + x_n + a_1 + \cdots + a_n)^{-s_n} \, dx_n = \frac{(x_1 + \cdots + x_{n-1} + a_1 + \cdots + a_{n-1} + 1 + a_n)^{-s_n+1}}{s_n - 1} \tag{3.8}$$

condition (3.5) yields

$$\int\limits_1^{+\infty} (x_1 + \cdots + x_n + a_1 + \cdots + a_n)^{-s_n} \, dx_n = \sum_{k_n \in \mathbb{N}} \binom{-s_n+1}{k_n} \frac{(1+a_n)^{k_n}}{s_n - 1} (x_1 + \cdots + x_{n-1} + a_1 + \cdots + a_{n-1})^{-s_n+1-k_n}. \tag{3.9}$$

If for $1 \le j \le n-1$, $\left(\sum_{i=j}^{n} \Re(s_i) - n + j - 1 + \sum_{i=j+1}^{n} k_i \right) > 0$ (3.10)

then inductively we find:

$$Y_{\underline{a}}(\underline{s}) = \sum_{\underline{k}=(k_2,\ldots,k_n)\in\mathbb{N}^{n-1}} \prod_{j=2}^{n} \binom{-\sum_{i=j}^n s_i + n - j + 1 - \sum_{i=j+1}^n k_i}{k_j} \times \frac{(1+a_j)^{k_j} (1+a_1)^{-\sum_{i=1}^n s_i + n - \sum_{i=2}^n k_i}}{(\sum_{i=1}^n s_i - n + \sum_{i=2}^n k_i)(\sum_{i=j}^n s_i - n + j - 1 + \sum_{i=j+1}^n k_i)}. \tag{3.11}$$

But, for all $2 \le j \le n$ we have: $(1+a_j)^{k_j} = \sum_{\substack{v_j \in \mathbb{N} \\ v_j \le k_j}} \binom{k_j}{v_j} a_j^{v_j}$ (3.12)

and $(1+a_1)^{-\sum_{i=1}^n s_i + n - \sum_{i=2}^n k_i} = \sum_{\substack{v_1 \in \mathbb{N} \\ v_1 \le (-\sum_{i=1}^n s_i + n - \sum_{i=2}^n k_i)}} \binom{-\sum_{i=1}^n s_i + n - \sum_{i=2}^n k_i}{v_1} a_1^{v_1}$ (3.13)

which yields

$$Y_{\underline{a}}(\underline{s}) = \sum_{\underline{k}=(k_2,\ldots,k_n)\in\mathbb{N}^{n-1}} \sum_{\substack{\underline{v}=(v_1,\ldots,v_n)\in\mathbb{N}^n \\ v_j \le k_j \ \forall 2\le j\le n;\ v_1 \le (-\sum_{i=1}^n s_i + n - \sum_{i=2}^n k_i)}} \prod_{j=2}^{n} \frac{A(\underline{s}) a_1^{v_1} a_j^{v_j}}{(\sum_{i=1}^n s_i - n + \sum_{i=2}^n k_i)(\sum_{i=j}^n s_i - n + j - 1 + \sum_{i=j+1}^n k_i)} \tag{3.14}$$

with $A(\underline{s}) = \binom{-\sum_{i=1}^n s_i + n - \sum_{i=2}^n k_i}{v_1} \prod_{j=2}^{n} \binom{-\sum_{i=j}^n s_i + n - j + 1 - \sum_{i=j+1}^n k_i}{k_j} \binom{k_j}{v_j}.$ (3.15)

Setting $\underline{s} = -\underline{N} = -(N_1, \ldots, N_n) \in \mathbb{N}^n$ yields (3.2) and ends the proof of Proposition 3.1. □

4. Proof of Theorem 1

The proof relies on the *Raabe* formula [7], which expresses the integral in terms of the sum.

Proposition 4.1.

(1) *Raabe formula:*

for all $\mathbf{\underline{s}} \in \mathbb{C}^n$, *outside the possible polar divisors of* $Y(\mathbf{\underline{s}})$, *we have:* $Y(\mathbf{\underline{s}}) = \int\limits_{\mathbf{\underline{t}} \in [0,1]^n} Z_{\mathbf{\underline{t}}}(\mathbf{\underline{s}}) \, d\mathbf{\underline{t}}$ (4.1)

where: $Z_{\mathbf{\underline{t}}}(\mathbf{\underline{s}}) = \sum_{\underline{m} \in \mathbb{N}^{*n}} \prod_{i=1}^{n} \frac{1}{((m_1+t_1) + \cdots + (m_i+t_i))^{s_i}}$

and $d\mathbf{\underline{t}}$ *is the Lebesgue measure on* $\mathbb{R}^n$.

(2) *For a fixed point* $\mathbf{\underline{N}} = (N_1, \ldots, N_n)$ *in* $\mathbb{N}^n$, *the maps* $\mathbf{\underline{a}} \mapsto Y_{\mathbf{\underline{a}}}(-\mathbf{\underline{N}})$ *and* $\mathbf{\underline{a}} \mapsto Z_{\mathbf{\underline{a}}}(-\mathbf{\underline{N}})$ *are polynomials in* $\mathbf{\underline{a}} = (a_1, \ldots, a_n) \in \mathbb{R}_+^n$.

Proof.

(1) Let $\underline{\mathbf{s}} \in \mathbb{C}^n$ be chosen in such a way that the integral function and the multiple zeta function are absolutely convergent. Thus, for $\underline{\mathbf{t}} \in \mathbb{R}_+^n$, we have:

$$\begin{aligned}\int_{[0,1]^n} Z_{\underline{\mathbf{t}}}(\underline{\mathbf{s}})\,\mathrm{d}\underline{\mathbf{t}} &= \int_{[0,1]^n} \sum_{\underline{\mathbf{m}} \in \mathbb{N}_*^n} \prod_{i=1}^n (t_1 + \cdots + t_i + m_1 + \cdots + m_i)^{-s_i}\,\mathrm{d}\underline{\mathbf{t}} \\ &= \sum_{\underline{\mathbf{m}}=(m_1,\ldots,m_n)\in\mathbb{N}_*^n} \int_{\prod_{i=1}^n [m_i, m_i+1]} \prod_{i=1}^n (t_1 + \cdots + t_i + m_1 + \cdots + m_i)^{-s_i}\,\mathrm{d}\underline{\mathbf{t}} \\ &= \int_{[1,+\infty[^n} \prod_{i=1}^n (x_1 + \cdots + x_i)^{-s_i}\,\mathrm{d}\underline{\mathbf{x}} = Y(\underline{\mathbf{s}}).\end{aligned}$$

This last equality, which is verified for all $\underline{\mathbf{s}} \in \mathbb{C}^n$, follows by analytic continuation outside the polar divisors.

(2) Follows from (3.2) combined with the *Raabe* formula. □

Lemma 4.1. *(See [6].)*

Let P and Q to be two polynomials in n variables linked by $P(\underline{\mathbf{a}}) = \int_{\underline{\mathbf{t}}\in[0,1]^n} Q(\underline{\mathbf{a}} + \underline{\mathbf{t}})\,\mathrm{d}\underline{\mathbf{t}}$. (4.2)

Write out $P(\underline{\mathbf{a}}) = P(a_1, \ldots, a_n) = \sum_{\underline{\mathbf{L}}} h_{\underline{\mathbf{L}}} \prod_{i=1}^n a_i^{L_i}$ (4.3)

where $h_{\underline{\mathbf{L}}} \in \mathbb{C}$ *and* $\underline{\mathbf{L}} = (L_1, \ldots, L_n) \in \mathbb{N}^n$ *ranges over a finite set of multi-index. Then*

$$Q(\underline{\mathbf{a}}) = Q(a_1, \ldots, a_n) = \sum_{\underline{\mathbf{L}}} h_{\underline{\mathbf{L}}} \prod_{i=1}^n B_{L_i}(a_i) \tag{4.4}$$

where the $B_{L_i}(a_i)$ *are the Bernoulli polynomials [2].*

Conversely, if Q is given by (4.4), then the relations (4.2) and (4.3) yield equivalent formulas for the polynomial P.

Proposition 4.2. *If we write out the polynomial* $Y_{\underline{\mathbf{a}}}(-\underline{\mathbf{N}})$ *as a sum of monomials,* $Y_{\underline{\mathbf{a}}}(-\underline{\mathbf{N}}) = \sum_{\underline{\mathbf{L}}} C_{\underline{\mathbf{L}}} \underline{\mathbf{a}}^{\underline{\mathbf{L}}}$ *with* $\underline{\mathbf{a}}^{\underline{\mathbf{L}}} = \prod_{i=1}^n a_i^{L_i}$ *and* $C_{\underline{\mathbf{L}}} = C_{\underline{\mathbf{L}}}(\underline{\mathbf{N}}) \in \mathbb{C}$.

Then $Z(-\underline{\mathbf{N}}) = \sum_{\underline{\mathbf{L}}} C_{\underline{\mathbf{L}}} B_{\underline{\mathbf{L}}}$ *where* $B_{\underline{\mathbf{L}}} = \prod_{i=1}^n B_{L_i}$ *is a product of Bernoulli numbers.*

More generally, for $\underline{\mathbf{a}} = (a_1, \ldots, a_n) \in \mathbb{R}_+^n$*, we have:* $Z_{\underline{\mathbf{a}}}(-\underline{\mathbf{N}}) = \sum_{\underline{\mathbf{L}}} C_{\underline{\mathbf{L}}} B_{\underline{\mathbf{L}}}(\underline{\mathbf{a}})$ *where* $B_{\underline{\mathbf{L}}}(\underline{\mathbf{a}}) = \prod_{i=1}^n B_{L_i}(a_i)$ *is a product of Bernoulli numbers.*

Proof. It follows from the above lemma, with $P(\underline{a}) = Y_{\underline{a}}(-\underline{\mathbf{N}})$ and $Q(\underline{a}) = Z_{\underline{a}}(-\underline{\mathbf{N}})$. □

4.1. Proof of Theorem 1

Relation (3.2) shows that for all $\underline{a} \in \mathbb{R}_+^n$

$$Y_{\underline{a}}(-\underline{N}) = \sum_{\underline{k}=(k_2,\ldots,k_n)\in\mathbb{N}^{n-1}} \sum_{\substack{\underline{v}=(v_1,\ldots,v_n)\in\mathbb{N}^n \\ v_j \le k_j\ \forall 2\le j\le n;\, v_1 \le (\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i)}} \prod_{j=2}^n \frac{(-1)^n A(-\underline{N}) a_1^{v_1} a_j^{v_j}}{(\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i)(\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i)} \tag{4.5}$$

with

$$A(-\underline{N}) = \binom{\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i}{v_1} \prod_{j=2}^n \binom{\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i}{k_j} \binom{k_j}{v_j} \tag{4.6}$$

and

$$T(\underline{N}) := \left\{ \underline{k} = (k_2, ..., k_n) \in \mathbb{N}^{n-1} : 0 \le k_j \le \sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i, \ \forall 2 \le j \le n \right\}. \tag{4.7}$$

Setting $\underline{a}^{\underline{v}} = a_1^{v_1} \prod_{j=2}^{n} a_j^{v_j}$ (4.8)

this gives

$$Y_{\underline{a}}(-\underline{N}) = \sum_{\underline{k}=(k_2,...,k_n)\in\mathbb{N}^{n-1}} \sum_{\substack{\underline{v}=(v_1,...,v_n)\in\mathbb{N}^n \\ v_j \le k_j \ \forall 2\le j\le n; v_1 \le (\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i)}} \prod_{j=2}^{n} \frac{(-1)^n A(-\underline{N})\underline{a}^{\underline{v}}}{(\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i)(\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i)}. \tag{4.9}$$

It follows from Proposition 4.2 that

$$Z(-\underline{N}) = \sum_{\underline{k}=(k_2,...,k_n)\in\mathbb{N}^{n-1}} \sum_{\substack{\underline{v}=(v_1,...,v_n)\in\mathbb{N}^n \\ v_j \le k_j \ \forall 2\le j\le n; v_1 \le (\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i)}} \prod_{j=2}^{n} \frac{(-1)^n A(-\underline{N}) B_{\underline{v}}}{(\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i)(\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i)} \tag{4.10}$$

with $B_{\underline{v}} = \prod_{j=1}^n B_{v_j}$ and B_{v_j} is the v_j-th Bernoulli number, which ends the proof of Theorem 1.

5. Double zeta values

In this part, we give an application of our main result for $n = 2$. We have:

$$Z(s_1, s_2) = \sum_{n_2 > n_1 > 0} \frac{1}{n_1^{s_1} n_2^{s_2}} = \sum_{(n_1,n_2)\in\mathbb{N}^{*2}} \frac{1}{n_1^{s_1}(n_1+n_2)^{s_2}}.$$

Thus, for all $\underline{a} = (a_1, a_2) \in \mathbb{R}_+^2$

$$Y_{\underline{a}}(\underline{s}) = Y_{\underline{a}}(s_1, s_2) = \int_1^{+\infty}\int_1^{+\infty} (x_1 + a_1)^{-s_1}(x_1 + x_2 + a_1 + a_2)^{-s_2} \mathrm{d}x_1 \mathrm{d}x_2.$$

Proposition 3.1 gives for $\underline{N} = (N_1, N_2) \in \mathbb{N}^2$

$$Y_{\underline{a}}(-\underline{N}) = \sum_{k=0}^{N_2+1} \sum_{v_1=0}^{N_1+N_2+2-k} \sum_{v_2=0}^{k} \frac{\binom{N_2+1}{k}\binom{N_1+N_2+2-k}{v_1}\binom{k}{v_2} a_1^{v_1} a_2^{v_2}}{(N_1+N_2+2-k)(N_2+1)}. \tag{5.1}$$

Proposition 4.1 shows that

$$Z(-\underline{N}) = Z(-N_1, -N_2) = \sum_{k=0}^{N_2+1} \sum_{v_1=0}^{N_1+N_2+2-k} \sum_{v_2=0}^{k} \frac{\binom{N_2+1}{k}\binom{N_1+N_2+2-k}{v_1}\binom{k}{v_2} B_{v_1} B_{v_2}}{(N_1+N_2+2-k)(N_2+1)} \tag{5.2}$$

with B_{v_1} and B_{v_2} are the v_1-th and v_1-th Bernoulli numbers (successively).

We give some values in the table below.

$\underline{N} = (N_1, N_2)$	$Z(-\underline{N})$	$\underline{N} = (N_1, N_2)$	$Z(-\underline{N})$	$\underline{N} = (N_1, N_2)$	$Z(-\underline{N})$
$(0,0)$	$\frac{1}{3}$	$(1,0)$	$\frac{1}{24}$	$(2,0)$	$\frac{-1}{120}$
$(0,1)$	$\frac{1}{12}$	$(1,1)$	$\frac{1}{360}$	$(2,1)$	$\frac{-1}{240}$
$(0,2)$	$\frac{1}{90}$	$(1,2)$	$\frac{-1}{240}$	$(2,2)$	$\frac{-1}{15120}$
$(0,3)$	$\frac{-1}{120}$	$(1,3)$	$\frac{-1}{560}$	$(2,3)$	$\frac{1}{504}$
$(0,4)$	$\frac{-1}{210}$	$(1,4)$	$\frac{1}{504}$	$(2,4)$	$\frac{29}{37800}$
$(0,5)$	$\frac{1}{252}$	$(1,5)$	$\frac{1}{504}$	$(2,5)$	$\frac{-1}{480}$
$(3,0)$	$\frac{-1}{240}$	$(4,0)$	$\frac{1}{252}$	$(5,0)$	$\frac{1}{504}$
$(3,1)$	$\frac{13}{10080}$	$(4,1)$	$\frac{1}{504}$	$(5,1)$	$\frac{-53}{30240}$
$(3,2)$	$\frac{1}{504}$	$(4,2)$	$\frac{-11}{15120}$	$(5,2)$	$\frac{-1}{480}$
$(3,3)$	$\frac{1}{50400}$	$(4,3)$	$\frac{-1}{480}$	$(5,3)$	$\frac{109}{133056}$
$(3,4)$	$\frac{-1}{480}$	$(4,4)$	$\frac{-1}{166320}$	$(5,4)$	$\frac{1}{264}$
$(3,5)$	$\frac{-557}{665280}$	$(4,5)$	$\frac{1}{264}$	$(5,5)$	$\frac{739}{181621440}$

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