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Functional analysis

Fréchet differentiability of the norm of L_p-spaces associated with arbitrary von Neumann algebras

Différentiabilité au sens de Fréchet de la norme d'un espace L_p associé à une algèbre de von Neumann arbitraire

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ABSTRACT

Let $\mathcal{M}$ be a von Neumann algebra and let $(\mathcal{L}_p(\mathcal{M}), \|\cdot\|_p)$, $1 \le p < \infty$ be Haagerup's L_p-space on $\mathcal{M}$. We prove that the differentiability properties of $\|\cdot\|_p$ are precisely the same as those of classical (commutative) L_p-spaces. Our main instruments are multiple operator integrals and singular traces.

RÉSUMÉ

Soit $\mathcal{M}$ une algèbre de von Neumann et soit $(\mathcal{L}_p(\mathcal{M}), \|\cdot\|_p)$, $1 \le p < \infty$ l'espace L_p de Haagerup sur $\mathcal{M}$. On montre que les propriétés de différentiabilité de $\|\cdot\|_p$ sont exactement les mêmes que celles obtenues sur les espaces L_p classiques (commutatifs). Les ingrédients principaux sont les opérateurs intégraux multiples et les traces singulières.

Version française abrégée

Soit $\mathcal{H}$ un espace de Hilbert et soit $B(\mathcal{H})$ l'espace des opérateurs bornés sur $\mathcal{H}$. Soit $\mathcal{M} \subseteq B(\mathcal{H})$ une algèbre de von Neumann. On note $\mathcal{L}_p(\mathcal{M})$ l'espace L_p de Haagerup sur l'algèbre $\mathcal{M}$ muni de la norme $\|\cdot\|_p$ (pour la définition et les propriétés de ces objets, nous renvoyons à [9,18]).

Le résultat qui suit résout la question relative à la Fréchet-différentiabilité de la norme sur $\mathcal{L}_p(\mathcal{M})$ suggérée par G. Pisier et Q. Xu dans leur article de présentation [14] (on renvoie à [12] pour les définitions et la terminologie utiles sur la différentiabilité en général). Dans le cas particulier où l'algèbre $\mathcal{M}$ est de type I, cette question est complètement résolue dans [15,20] (voir [5,17] pour le cas commutatif). Le cas général (même lorsque $\mathcal{M}$ est semifinie) nécessite de nouvelles idées.

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Théorème 0.1. *La fonction* $H \mapsto \|H\|_p^p$, $H \in \mathcal{L}_p(\mathcal{M})$, $1 < p < \infty$, *est*

(i) *indéfiniment Fréchet différentiable lorsque* p *est un entier pair ;*
(ii) $(p-1)$ *fois Fréchet différentiable lorsque* p *est un entier impair ;*
(iii) $[p]$ *fois Fréchet différentiable lorsque* p *n'est pas un entier.*

Le résultat du Théorème 0.1 est optimal.

Lorsque $\mathcal{M}$ est une algèbre de von Neumann semifinie sur un espace de Hilbert $\mathcal{H}$ et que τ est une trace normale semifinie et fidèle sur $\mathcal{M}$, on note $(L_p(\mathcal{M},\tau), \|\cdot\|_p)$ l'espace L_p non commutatif classique associé à $(\mathcal{M},\tau)$, muni de la norme $\|\cdot\|_p$ donnée par $\|x\|_p := \tau(|x|^p)^{1/p}$, $x \in L_p(\mathcal{M},\tau)$.

Il est bien connu (voir par exemple [9, Theorem 2.1], [18, p. 62]) que les espaces $(\mathcal{L}_p(\mathcal{M}), \|\cdot\|_p)$ et $(L_p(\mathcal{M},\tau), \|\cdot\|_p)$ sont isométriquement isomorphes pour tout $1 \le p < \infty$. Par conséquent, le résultat du Théorème 0.1 décrit également les propriétés de différentiabilité des espaces L_p non commutatifs $(L_p(\mathcal{M},\tau), \|\cdot\|_p)$, $1 \le p < \infty$, ce qui renforce significativement les résultats antérieurs de [5,15,17,20].

1. Haagerup's L_p-spaces

In this section we recall the construction of noncommutative L_p-spaces associated with an arbitrary von Neumann algebra. We use Haagerup's definition [9], and Terp's exposition of the subject [18]. The basics on von Neumann algebras and Tomita's modular theory may be found in [10].

Let $\mathcal{N}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ. An unbounded densely defined operator is said to be affiliated with $\mathcal{N}$ if it commutes with all elements in the commutant of $\mathcal{N}$. A closed densely defined operator A affiliated with $\mathcal{N}$ is called τ-measurable if $\tau(E_{|A|}(n,\infty)) \to 0$ as $n \to \infty$, where $E_{|A|}(n,\infty)$ is the spectral projection of the self-adjoint operator $|A| = (A^*A)^{\frac{1}{2}}$ corresponding to the interval (n,∞). By $\mathcal{S}(\mathcal{N},\tau)$ we denote the complete topological $*$-algebra of all τ-measurable operators equipped with the measure topology (see [8,11]).

Let $\mathcal{M}$ be an arbitrary von Neumann algebra with a faithful normal semifinite weight ϕ_0. We consider the one-parameter modular automorphism group $\sigma^{\phi_0} = \{\sigma_t^{\phi_0}\}_{t\in\mathbb{R}}$ (associated with ϕ_0) on $\mathcal{M}$ and obtain a semifinite crossed product von Neumann algebra $\mathcal{N} := \mathcal{M} \rtimes_{\sigma^{\phi_0}} \mathbb{R}$, which admits the canonical semifinite trace τ and a trace-scaling dual action $\theta = \{\theta_s\}_{s\in\mathbb{R}}$ such that $\tau \circ \theta_s = e^{-s}\tau$ for all $s \in \mathbb{R}$.

The original von Neumann algebra $\mathcal{M}$ can be identified with a θ-invariant von Neumann subalgebra $\mathcal{L}_\infty(\mathcal{M})$ of $\mathcal{N}$. For $1 \le p < \infty$, the noncommutative L_p-space $\mathcal{L}_p(\mathcal{M})$ is defined as follows:

$$\mathcal{L}_p(\mathcal{M}) := \left\{A \in \mathcal{S}(\mathcal{N},\tau) : \theta_s(A) = e^{-\frac{s}{p}} A \text{ for all } s \in \mathbb{R}\right\}.$$

It is known from [18, Part II, Theorem 7] that there is a linear bijection $\psi \mapsto A_\psi$ between the predual space $\mathcal{M}_*$ and $\mathcal{L}_1(\mathcal{M})$. Due to this correspondence, we may define the trace $\mathrm{tr} : \mathcal{L}_1(\mathcal{M}) \to \mathbb{C}$ as follows:

$$\mathrm{tr}(A_\psi) := \psi(1), \quad A_\psi \in \mathcal{L}_1(\mathcal{M}).$$

Given any $A \in \mathcal{L}_p(\mathcal{M})$, $1 \le p < \infty$, we have the polar decomposition $A = U|A|$, where $|A|$ is a positive operator in $\mathcal{L}_p(\mathcal{M})$ and U is a partial isometry contained in $\mathcal{M}$. It is established in [18, Proposition 12] that $|A|^p \in \mathcal{L}_1(\mathcal{M})$. Thus, we can define a Banach norm (see [18, Corollary 27]) on $\mathcal{L}_p(\mathcal{M})$ by setting:

$$\|A\|_p := \mathrm{tr}\big(|A|^p\big)^{\frac{1}{p}}, \quad A \in \mathcal{L}_p(\mathcal{M}).$$

It is well known (see, e.g., [9, Theorem 2.1], [18, p. 62]), that if $\mathcal{M}$ is a semifinite von Neumann algebra equipped with a normal faithful semifinite trace τ, then the space $\mathcal{L}_p(\mathcal{M})$ is isometrically isomorphic to the classical noncommutative L_p-space $L_p(\mathcal{M},\tau)$ (we refer to [14] for definitions of L_p-spaces and all relevant properties).

2. Main theorem

The following result resolves the question concerning the Fréchet differentiability of the norms of noncommutative L_p-spaces associated with an arbitrary von Neumann algebra $\mathcal{M}$, suggested by G. Pisier and Q. Xu in their survey [14] (we refer to [12] for all relevant definitions and terminology concerning differentials of abstract functions). In the special case when the algebra $\mathcal{M}$ is of type I, this question is fully resolved in [15,20] (see [5,17] for the commutative result).

Theorem 2.1. *The function* $H \mapsto \|H\|_p^p$, $H \in \mathcal{L}_p(\mathcal{M})$, $1 < p < \infty$, *is*

(i) *infinitely many times Fréchet differentiable, whenever* p *is an even integer;*
(ii) $(p-1)$*-times Fréchet differentiable, whenever* p *is an odd integer;*
(iii) $[p]$*-times Fréchet differentiable, whenever* p *is not an integer.*

The result of Theorem 2.1 is sharp. The Fréchet derivatives for the function $H \mapsto \|H\|_p^p$, $H \in \mathcal{L}_p(\mathcal{M})$ are given in Definition 6.1 below and the Taylor expansion is proved in Theorem 6.1.

In the special case when $\mathcal{M}$ is a semifinite von Neumann algebra equipped with a normal faithful semifinite trace τ, the result of Theorem 2.1 also describes the differentiability properties of the classical noncommutative L_p-spaces $(L_p(\mathcal{M}, \tau), \|\cdot\|_p)$, $1 \le p < \infty$ and thus substantially strengthens results from [5,15,17,20].

3. Classical noncommutative $L_p(\mathcal{N}, \tau)$ and $L_{p,\infty}(\mathcal{N}, \tau)$

The notions of the distribution function $n_{|A|}$, and that of the singular value function $\mu(A) : t \mapsto \mu(t, A)$ for $A \in \mathcal{S}(\mathcal{N}, \tau)$, are defined as follows:

$$n_{|A|}(t) := \tau\big(E_{|A|}(t, \infty)\big), \quad t \in \mathbb{R}; \qquad \mu(t, A) := \inf\big\{s \ge 0 : n_{|A|}(s) \le t\big\}, \quad t \ge 0.$$

The classical noncommutative space $L_p(\mathcal{N}, \tau)$, $1 \le p < \infty$ is defined as follows:

$$L_p(\mathcal{N}, \tau) := \big\{A \in \mathcal{S}(\mathcal{N}, \tau) : \ \mu\big(|A|\big) \in L_p(0, \infty)\big\}, \quad \|A\|_p := \big\|\mu\big(|A|\big)\big\|_p,$$

where $(L_p(0, \infty), \|\cdot\|_p)$ is the usual Lebesgue space.

The space $L_{p,\infty}(\mathcal{N}, \tau)$, $1 \le p < \infty$ is the set of all $A \in \mathcal{S}(\mathcal{N}, \tau)$ such that:

$$\|A\|'_{p,\infty} := \sup_{t \ge 0} t^{\frac{1}{p}} \mu(t, A) < +\infty.$$

For $1 \le p < \infty$, the space $L_{p,\infty}(\mathcal{N}, \tau)$ equipped with the quasi-norm $\|\cdot\|'_{p,\infty}$ is a quasi-Banach space. For $1 < p < \infty$, there exists a norm $\|\cdot\|_{p,\infty}$ on $L_{p,\infty}(\mathcal{N}, \tau)$ given by

$$\|A\|_{p,\infty} := \sup_{t > 0} t^{\frac{1}{p} - 1} \int_0^t \mu(s, A) \mathrm{d}s, \quad A \in L_{p,\infty}(\mathcal{N}, \tau),$$

which is equivalent to $\|\cdot\|'_{p,\infty}$ (see e.g. [11, Example 2.6.10]).

4. Traces on $L_{1,\infty}(\mathcal{N}, \tau)$

A trace φ on $L_{1,\infty}(\mathcal{N}, \tau)$ is a linear functional, which is unitarily invariant, i.e. $\varphi : L_{1,\infty}(\mathcal{N}, \tau) \to \mathbb{C}$ satisfies $\varphi(UAU^*) = \varphi(A)$ for all $A \in L_{1,\infty}(\mathcal{N}, \tau)$ and all unitaries $U \in \mathcal{N}$.

A trace φ on $L_{1,\infty}(\mathcal{N}, \tau)$ is said to be normalized if $\varphi(A) = 1$, for every $0 \le A \in L_{1,\infty}(\mathcal{N}, \tau)$ with $\mu(t, A) = t^{-1}$, $t > 0$. We refer to [11] for the theory of traces on ideals in semifinite von Neumann algebras.

Lemma 4.1. *Let $(\mathcal{M}, \phi_0)$ be a von Neumann algebra equipped with a faithful normal semifinite weight; let $\mathcal{N} = \mathcal{M} \rtimes_{\sigma^{\phi_0}} \mathbb{R}$ be the crossed product von Neumann algebra equipped with the faithful normal semifinite trace τ. For every normalized trace φ on $L_{1,\infty}(\mathcal{N}, \tau)$ we have that*

$$\varphi(A) = \mathrm{tr}(A), \quad A \in \mathcal{L}_1(\mathcal{M}).$$

5. Multiple operator integrals

Let $k \in \mathbb{N}$ and a family $(X_j, \|\cdot\|_{X_j})$ $1 \le j \le k$ of Banach spaces be given. By $X_1 \times \ldots \times X_k$, we denote the Cartesian product of the Banach spaces $X_1, \ldots, X_k$. If $X = X_1 = \ldots = X_k$, then we write $X^{\times k} = X_1 \times \ldots \times X_k$. Let $(Y, \|\cdot\|_Y)$ be a Banach space. A multilinear operator $T : X_1 \times \ldots \times X_k \to Y$ is said to be bounded if

$$\|T\|_{X_1 \times \ldots \times X_k \to Y} := \sup\big\{\big\|T(x_1, \ldots, x_k)\big\|_Y : \|x_j\|_{X_j} \le 1, \ 1 \le j \le k\big\} < \infty.$$

The set of all bounded multilinear operators $T : X_1 \times \ldots \times X_k \to Y$ is denoted by $\mathcal{B}(X_1 \times \ldots \times X_k, Y)$.

Let $\mathcal{N}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ.

Définition 5.1. Let $k \in \mathbb{N}$ and let $1 \le p_j \le \infty$, $1 \le j \le k$ be such that $0 \le \sum_{j=1}^k \frac{1}{p_j} \le 1$. Let a bounded Borel function $\phi : \mathbb{R}^{k+1} \to \mathbb{C}$ be fixed. Let $H_0, \ldots, H_k$ be self-adjoint operators affiliated with $\mathcal{N}$. Suppose that for all $V_j \in L_{p_j}(\mathcal{N}, \tau)$, $1 \le j \le k$, for every $m \in \mathbb{N}$ the series

$$S_{\phi,m}(V_1, \ldots, V_k) := \sum_{l_0, \ldots, l_k \in \mathbb{Z}} \phi\left(\frac{l_0}{m}, \ldots, \frac{l_k}{m}\right) E_{H_0}\left(\left[\frac{l_0}{m}, \frac{l_0 + 1}{m}\right)\right) \prod_{j=1}^k V_j E_{H_j}\left(\left[\frac{l_j}{m}, \frac{l_j + 1}{m}\right)\right)$$

converges in the norm of $L_p(\mathcal{N},\tau)$, where $\frac{1}{p}=\sum_{j=1}^{k}\frac{1}{p_j}$ and

$$(V_1,\dots,V_k)\mapsto S_{\phi,m}(V_1,\dots,V_k),\quad m\in\mathbb{N}$$

is a sequence from $\mathcal{B}(L_{p_1}(\mathcal{N},\tau)\times\ldots\times L_{p_k}(\mathcal{N},\tau),L_p(\mathcal{N},\tau))$. If the sequence $\{S_{\phi,m}\}_{m\geq 1}$ converges pointwise to some multilinear operator $T_\phi^{H_0,\dots,H_k}$, then, according to the Banach–Steinhaus theorem (see, e.g., [19]), $\{S_{\phi,m}\}_{m\geq 1}$ is uniformly bounded and $T_\phi^{H_0,\dots,H_k}\in\mathcal{B}(L_{p_1}(\mathcal{N},\tau)\times\ldots\times L_{p_k}(\mathcal{N},\tau),L_p(\mathcal{N},\tau))$. In this case, the operator $T_\phi^{H_0,\dots,H_k}$ is called a *multiple operator integral*.

The definition above differs from the original approach to multiple operator integration developed in [2–4] and various earlier modifications in [1,6,7,13]. However, for a large class of functions ϕ, all these definitions coincide (see, e.g., [16, Lemma 3.5] or [15, Section 1], which contains a detailed comparison of these definitions).

Using results from [16], we extend the multiple operator integral defined above to the operator (denoted again by $T_\phi^{H_0,\dots,H_k}$) from $\mathcal{B}(L_{p,\infty}(\mathcal{N},\tau)^{\times k},L_{\frac{p}{p-1},\infty}(\mathcal{N},\tau))$, $1<p<\infty$.

6. Taylor expansion

Observe that the norm of $\mathcal{L}_p(\mathcal{M})$, when p is even, is infinitely many times differentiable.

Observe also that it is sufficient to establish the assertion of Theorem 2.1 only for self-adjoint operators. Indeed, introducing the von Neumann algebra $\mathcal{A}:=\mathcal{M}\bar{\otimes}M_2$, where M_2 is the von Neumann algebra of all 2×2 complex matrices, we have that $\mathcal{L}_p(\mathcal{M})$ isometrically embeds into the (real) subspace of all self-adjoint operators from $\mathcal{L}_p(\mathcal{A})$.

Let f be a compactly supported function such that $f(t)=|t|$, in some neighborhood of zero and f is smooth except at zero. Let $1<p<\infty$ and $m\in\mathbb{N}$ be such that $p\in(m,m+1]$.

Définition 6.1. Let $H=H^*\in L_{p,\infty}(\mathcal{N},\tau)$, $p\in(m,m+1]$. Let φ be a normalized positive trace on $L_{1,\infty}(\mathcal{N},\tau)$. For $V_1,\dots,V_m\in L_{p,\infty}(\mathcal{N},\tau)$, we define

$$\delta^H_{k,p,\varphi}(V_1,\dots,V_k):=\begin{cases}\varphi(V_1\cdot(f^p)'(H)), & k=1\\ \frac{1}{k!}\sum_{\pi\in\mathfrak{S}_k}\varphi(V_{\pi(1)}\cdot T^{H,\cdots,H}_{((f^p)')^{[k-1]}}(V_{\pi(2)},\dots,V_{\pi(k)})), & 2\leq k\leq m,\end{cases}$$

where $\mathfrak{S}_k$ is a set of permutations of $\{1,\dots,k\}$, and $((f^p)')^{[k-1]}$ is $(k-1)$-th divided difference of the function $(f^p)'$.

We prove that the multilinear functional $\delta^H_{k,p,\varphi}$ is bounded on $L_{p,\infty}(\mathcal{N},\tau)^{\times k}$, $1\leq k\leq m$. It follows from [8, Lemma 1.7] that $\|A\|_p=\|A\|'_{p,\infty}$ for all $A\in\mathcal{L}_p(\mathcal{M})$, and, therefore, $\mathcal{L}_p(\mathcal{M})$ is a closed linear subspace in $L_{p,\infty}(\mathcal{N},\tau)$. In Theorem 6.1 below, we show that the restriction of $\delta^H_{k,p,\varphi}$ to the subspace $\mathcal{L}_p(\mathcal{M})^{\times k}$ is a k-th Fréchet derivative for the function $H\mapsto\|H\|_p^p$, $H\in\mathcal{L}_p(\mathcal{M})$.

Theorem 6.1. *Let $p\in(m,m+1]$. If $H=H^*\in\mathcal{L}_p(\mathcal{M})$, then the restriction of $\delta^H_{k,p,\varphi}$ to $\mathcal{L}_p(\mathcal{M})^{\times k}$ is a symmetric multilinear bounded functional on $\mathcal{L}_p(\mathcal{M})^{\times k}$ for every $1\leq k\leq m$. Moreover, for $H=H^*$, $V=V^*\in\mathcal{L}_p(\mathcal{M})$, we have:*

$$\|H+V\|_p^p-\|H\|_p^p-\sum_{k=1}^{m}\frac{1}{k}\delta^H_{k,p,\varphi}(\underbrace{V,\dots,V}_{k\text{-times}})=O\big(\|V\|_p^p\big).$$

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