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Complex analysis

On the class of bi-univalent functions

Sur la classe des fonctions bi-univalentes

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ABSTRACT

In an attempt to answer the question raised by A.W. Goodman, we obtain a covering theorem, a distortion theorem, a growth theorem, the radius of convexity and an argument estimate of $f'(z)$ for functions of the class σ of *bi-univalent* functions.

RÉSUMÉ

Dans une tentative de répondre à une question posée par A.W. Goodman, nous obtenons des théorèmes de surjectivité, de déformation et de croissance, ainsi qu'une estimation du rayon de convexité et de l'argument de $f'(z)$ pour une fonction f dans la classe σ des fonctions *bi-univalentes*.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n, \quad z\in\mathbb{U}, \tag{1.1}$$

which are *analytic* in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. Further, by $\mathcal{S}$ we denote the class of all functions in $\mathcal{A}$ that are *univalent* in $\mathbb{U}$ (for more details on univalent functions, one may refer to [4]).

Obviously, every function $f\in\mathcal{S}$ has an inverse f^{-1}, defined by $f^{-1}(f(z))=z$, $z\in\mathbb{U}$, and $f(f^{-1}(w))=w$, $|w|<r_0(f)$, $r_0(f)\geq\frac{1}{4}$. Moreover, it is easy to see that the inverse function has the series expansion of the form:

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$$f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \ldots, \quad w \in \mathbb{U}.$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$, and let σ denote the class of *bi-univalent* functions in $\mathbb{U}$ of the form (1.1). For examples of bi-univalent functions, see the recent work of Srivastava et al. [14], and many other papers like [1,5,8–11,13,15–17].

We emphasize that, as in the class $\mathcal{S}$ of normalized univalent functions, the convex combination of two functions of class σ need not to be bi-univalent. For example, the functions $f_1(z) = \frac{z}{1-z}$ and $f_2(z) = \frac{z}{1+\mathrm{i}z}$ are *bi-univalent* but their sum $f_1 + f_2$ is not even univalent, as its derivative vanishes at $\frac{1}{2}(1+\mathrm{i})$. However, the class σ is preserved under a number of elementary transformations. In this regard, we give a result in Section 2.

Lewin [10] investigated the class σ of *bi-univalent functions* and obtained a bound

$$|a_2| < 1.51. \tag{1.2}$$

Motivated by the work of Lewin [10], Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$. Brannan and Taha [3] introduced the notions of *strongly bi-starlike functions of order* α and *strongly bi-convex functions of order* α and obtained coefficient bounds for $|a_2|$ and $|a_3|$. Following Brannan and Taha [3], many researchers [1,5,8–11,13,15–17] have recently studied several subclasses of σ and obtained coefficient bounds for $|a_2|$ and $|a_3|$.

In a survey article, A.W. Goodman [6, pages 170–172, question number 2] raised the question that $\max|a_n|$, $\max|f'(z)|$, $\max(\arg f'(z))$, etc. are not known for the functions in the class σ.

In the present article, we answer the above question raised by A.W. Goodman [6]. Also, we give the covering theorem for bi-univalent functions, which merely states that the range of each function in the class σ must contain a disk of minimum radius $\frac{1}{3.02}$. Further, we obtain the distortion theorem, the growth theorem and the radius of convexity for the functions of the class σ.

2. Covering theorem for bi-univalent functions

In this section of the paper, first we will show that the class σ is preserved under a number of elementary transformations, and we will give a covering theorem for the class σ. We begin with the partial list of elementary transformations under which the class σ is preserved.

Lemma 2.1. *The class σ is preserved under the following transformations:*

1. ***Rotation**: If $f \in \sigma$, $\theta \in \mathbb{R}$, and $g(z) = e^{-\mathrm{i}\theta} f(e^{\mathrm{i}\theta}z)$, then $g \in \sigma$;*
2. ***Dilation**: If $f \in \sigma$, $0 < r < 1$, and $g(z) = \frac{1}{r} f(rz)$, then $g \in \sigma$;*
3. ***Conjugation**: If $f \in \sigma$ and $g(z) = \overline{f(\bar{z})}$, then $g \in \sigma$;*
4. ***Disk automorphism**: If $f \in \sigma$, $\zeta \in \mathbb{U}$, and $g(z) = \frac{f(\frac{z+\zeta}{1+\bar{\zeta}z}) - f(\zeta)}{(1-|\zeta|^2)f'(\zeta)}$, then $g \in \sigma$;*
5. ***Omitted value transformation**: If $f \in \sigma$ with $f(z) \neq w$ for all $z \in \mathbb{U}$, and $g(z) = \frac{wf(z)}{w-f(z)}$, then $g \in \sigma$.*

Proof. The proofs of 1. to 5. are fairly straight forward, and hence we omit the details involved. But for the sake of completeness, we prove the bi-univalency of the omitted value transformation.

In the case of omitted value transformation, the function $g = T \circ f$, with $T(z) = \frac{wz}{w-z}$, where T is a fractional linear transformation, which is univalent and invertible. Since $f \in \sigma$, then $g = T \circ f \in \sigma$, with $g^{-1} = f^{-1} \circ T^{-1}$. $\square$

As the *Koebe function* $f(z) = \frac{z}{(1-z)^2}$ is not a member of the class σ and it plays the role of extremal functions in the class $\mathcal{S}$, the corresponding extremal properties of the class $\mathcal{S}$ is bound to change. As a first result in this direction, we obtain the covering theorem for the class σ. Interestingly, we found that the minimum radius of the disk contained in the range of functions of class σ is increased from $\frac{1}{4}$ to $\frac{1}{3.02}$, which is shown as follows:

Theorem 2.1 *(Covering theorem). The range of every function of the class σ contains the disk $\{w \in \mathbb{C} : |w| \leq \frac{1}{3.02}\}$.*

Proof. If $f \in \sigma$ omits the value $w \in \mathbb{C}$, then

$$h(z) = \frac{wf(z)}{w - f(z)} = z + \left(a_2 + \frac{1}{w}\right)z^2 + \ldots, \quad z \in \mathbb{U},$$

is analytic and bi-univalent in $\mathbb{U}$. Now, combining the inequality (1.2) with $|a_2 + \frac{1}{w}| \leq 1.51$, we obtain that $|w| \geq \frac{1}{3.02}$. $\square$

Remarks 2.1. 1. We emphasize that the above property is a necessary condition for a function to be bi-univalent. Also, we note that the famous Koebe function is not bi-univalent, since it does not satisfy the above property. In fact, the maximum of radius of the disk contained in the range of the Koebe function is $\frac{1}{4}$.

2. The bi-univalency condition is necessary to be assumed in the above theorem, as the function $f_n(z) = \frac{1}{n}(e^{nz} - 1)$ omits the value $-\frac{1}{n}$, which is as close to zero as n tends to infinity, and this function is not bi-univalent.

3. Distortion and rotation theorems

Lewin's inequality (1.2) has further implications in the geometric theory of bi-univalent functions. One important consequence is the distortion theorem, which provides non sharp upper and lower bounds for $|f'(z)|$ as f ranges over the class σ. The following theorem gives a basic estimate that leads to the distortion theorem and related results:

Theorem 3.1. *For each function $f \in \sigma$, we have:*

$$\left| \frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2} \right| < \frac{3.02r}{1-r^2}, \quad |z| = r < 1. \tag{3.1}$$

Proof. For a given function $f \in \sigma$ and a fixed $\zeta \in \mathbb{U}$, let perform a disk automorphism to define the function F by

$$F(z) = \frac{f(\frac{z+\zeta}{1+\bar{\zeta}z}) - f(\zeta)}{(1-|\zeta|^2)f'(\zeta)} = z + A_2(\zeta)z^2 + \dots, \quad z \in \mathbb{U}.$$

Then, according to Lemma 2.1 we have $F \in \sigma$, and a simple computation gives

$$A_2(\zeta) = \frac{1}{2}\left(\left(1 - |\zeta|^2\right) \frac{f''(\zeta)}{f'(\zeta)} - 2\bar{\zeta} \right), \tag{3.2}$$

and in view of Lewin's work [10] we obtain that $|A_2(\zeta)| < 1.51$. Therefore, by using the above bound for $A_2(\zeta)$ in Eq. (3.2) and replacing ζ by z, we finally get the inequality (3.1). □

We are now ready to prove the following distortion theorem for the functions in the class σ of bi-univalent functions:

Theorem 3.2 *(Distortion theorem). For each $f \in \sigma$, we have:*

$$\frac{(1-r)^{0.51}}{(1+r)^{2.51}} < \left|f'(z)\right| < \frac{(1+r)^{0.51}}{(1-r)^{2.51}}, \quad |z| = r < 1. \tag{3.3}$$

Proof. From inequality (3.1), we get:

$$\frac{2r^2 - 3.02\,r}{1-r^2} < \operatorname{Re} \frac{zf''(z)}{f'(z)} < \frac{2r^2 + 3.02\,r}{1-r^2}, \quad |z| = r < 1. \tag{3.4}$$

Because $|f'(z)| \neq 0$ and $f'(0) = 1$, we can choose a single valued branch of $\log f'(z)$ that vanishes at the origin. Now, we observe that

$$\operatorname{Re} \frac{zf''(z)}{f'(z)} = r\frac{\partial}{\partial r} \operatorname{Re}\left[\log f'(z)\right], \quad z = r\mathrm{e}^{\mathrm{i}\theta}.$$

Therefore, using the above identity in (3.4) we obtain

$$\frac{2r - 3.02}{1-r^2} < \frac{\partial}{\partial r} \log\left|f'(z)\right| < \frac{2r + 3.02}{1-r^2}, \quad z = r\mathrm{e}^{\mathrm{i}\theta}. \tag{3.5}$$

Holding θ fixed, if we integrate the inequality (3.5) with respect to r from 0 to R, a simple computation yields the inequality:

$$\log \frac{(1-R)^{0.51}}{(1+R)^{2.51}} < \log\left|f'\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\right| < \log \frac{(1+R)^{0.51}}{(1-R)^{2.51}}. \tag{3.6}$$

Finally, by exponentiating (3.6), we get (3.3). □

Here we would like to point out that the upper and lower bounds of the distortion factor $|f'(z)|$ for the class σ is obtained by considering only the real part of the inequality (3.1) in Theorem 3.1. Instead of this, if we consider the imaginary part we can obtain bound for the rotation factor $|\arg f'(z)|$. Thus, the following rotation theorem holds:

Theorem 3.3 *(Rotation theorem). For each $f \in \sigma$, we have:*

$$\left|\arg f'(z)\right| < 1.51 \log \frac{1+r}{1-r}, \quad |z| = r < 1.$$

Proof. From (3.1), considering the imaginary part alone, we get:

$$\frac{-3.02\,r}{1-r^2} < \operatorname{Im}\frac{zf''(z)}{f'(z)} < \frac{3.02\,r}{1-r^2}, \quad |z| = r < 1. \tag{3.7}$$

Because $|f'(z)| \neq 0$ and $f'(0) = 1$, we can choose a single valued branch of $\log f'(z)$ that vanishes at the origin. Now, we observe that

$$\operatorname{Im}\frac{zf''(z)}{f'(z)} = r\frac{\partial}{\partial r}\arg f'\big(re^{i\theta}\big), \quad z = re^{i\theta}.$$

Therefore, using the above identity in (3.7), we get:

$$\frac{-3.02}{1-r^2} < \frac{\partial}{\partial r}\arg f'\big(re^{i\theta}\big) < \frac{3.02}{1-r^2}, \quad z = re^{i\theta}. \tag{3.8}$$

Holding θ fixed and integrating the inequality (3.8) with respect to r from 0 to R, a simple computation yields the desired inequality. □

We notice that Theorems 3.2 and 3.3 answer some of the questions raised by A.W. Goodman [6].

Inequality (3.1) has further implications on the radius of convexity also. That is, for functions in the class σ, the upper bound for the radius of convexity is increased from $2-\sqrt{3} = 0.27\ldots$ to $1.51-\sqrt{1.2801} = 0.38\ldots$. Thus, the following corollary gives an estimate for the radius of convexity for functions in the class σ:

Corollary 3.1. *For every positive number $\rho < 1.51-\sqrt{1.2801} = 0.38\ldots$, each function $f \in \sigma$ maps the disk $|z| < \rho$ onto a convex domain.*

Proof. In view of inequality (3.1), we have the estimate:

$$\operatorname{Re}\left[1+\frac{zf''(z)}{f'(z)}\right] \geq \frac{1-3.02\,r+r^2}{1-r^2}, \quad |z| = r < 1.$$

But $1-3.02\,r+r^2 > 0$ for $r < 1.51-\sqrt{1.2801} = 0.38\ldots$, and therefore f maps such a disk $|z| < r$ onto a convex domain, which proves our result. □

4. Growth theorem

The distortion result given by Theorem 3.2 can be applied to obtain the lower and upper bounds for $|f(z)|$. To prove the result, we need the following lemma of Privalov [12] (see also [7, page 67]):

Lemma 4.1. *(See [12].) Suppose that $f \in \mathcal{S}$ and that for $0 \leq r < 1$, we have $m'(r) \leq |f'(z)| \leq M'(r)$, where $m'(r)$ and $M'(r)$ are real valued functions of r in $[0,1)$. Then,*

$$\int_0^r m'(t)\,dt \leq \big|f(z)\big| \leq \int_0^r M'(t)\,dt.$$

We are now ready to prove the following growth theorem:

Theorem 4.1 *(Growth theorem). For each $f \in \sigma$, we have:*

$$\frac{1}{3.02}\left[1-\left(\frac{1-r}{1+r}\right)^{1.51}\right] \leq \big|f(z)\big| \leq \frac{1}{3.02}\left[\left(\frac{1+r}{1-r}\right)^{1.51}-1\right], \quad |z| = r < 1. \tag{4.1}$$

Proof. Let $f \in \sigma$ and fix $z = re^{i\theta}$, with $0 < r < 1$. According to Theorem 3.2, we could choose

$$m'(r) = \frac{(1-r)^{0.51}}{(1+r)^{2.51}} \quad \text{and} \quad M'(r) = \frac{(1+r)^{0.51}}{(1-r)^{2.51}},$$

and using the fact that $f \in \sigma \subset \mathcal{S}$, we can apply the Lemma 4.1 to get

$$\int_0^r \frac{(1-\rho)^{0.51}}{(1+\rho)^{2.51}}\,d\rho \leq \big|f(z)\big| \leq \int_0^r \frac{(1+\rho)^{0.51}}{(1-\rho)^{2.51}}\,d\rho.$$

Since the functions $\Phi_1(\rho) = -\frac{1}{3.02}(\frac{1-\rho}{1+\rho})^{1.51}$ and $\Phi_2(\rho) = \frac{1}{3.02}(\frac{1+\rho}{1-\rho})^{1.51}$ are primitives for those that are integrated in the left-hand and the right-hand side of the above inequality, respectively, a simple computation gives the double inequality (4.1). □

By combining the growth and distortion theorems, the following useful inequality is obtained:

Theorem 4.2 *(Combined growth and distortion theorem). For each $f \in \sigma$, we have:*

$$\frac{3.02\,r}{\frac{(1+r)^{2.51}}{(1-r)^{0.51}} - 1 + r^2} \le \left|\frac{zf'(z)}{f(z)}\right| \le \frac{3.02\,r}{1 - r^2 - \frac{(1-r)^{2.51}}{(1+r)^{0.51}}}, \quad 0 < |z| = r < 1.$$

Proof. For a given $\zeta \in \mathbb{U}$, let consider the function F defined by the disk automorphism:

$$F(z) = \frac{f(\frac{z+\zeta}{1+\bar{\zeta}z}) - f(\zeta)}{(1-|\zeta|^2)f'(\zeta)} = z + A_2(\zeta)z^2 + \ldots, \quad z \in \mathbb{U},$$

which is a member of the class σ. By applying Theorem 4.1 to $F(-\zeta)$, we get:

$$\frac{1}{3.02}\left[1 - \left(\frac{1-|\zeta|}{1+|\zeta|}\right)^{1.51}\right] \le \left|F(-\zeta)\right| \le \frac{1}{3.02}\left[\left(\frac{1+|\zeta|}{1-|\zeta|}\right)^{1.51} - 1\right], \quad \zeta \in \mathbb{U},$$

hence,

$$\frac{1-|\zeta|^2}{3.02\,|\zeta|}\left[1 - \left(\frac{1-|\zeta|}{1+|\zeta|}\right)^{1.51}\right] \le \left|\frac{f(\zeta)}{\zeta f'(\zeta)}\right| \le \frac{1-|\zeta|^2}{3.02\,|\zeta|}\left[\left(\frac{1+|\zeta|}{1-|\zeta|}\right)^{1.51} - 1\right]. \tag{4.2}$$

From (4.2), by changing ζ into z, we get the desired result. □

5. Concluding remarks

In the case when Brannan and Clunie's conjecture [2] could be proved affirmatively, our theorems will have better estimates, as follows.

1. **Covering theorem**: the range of every function of the class σ contains the disk $\{w \in \mathbb{C} : |w| < \frac{1}{2\sqrt{2}}\}$;
2. For each function $f \in \sigma$, we have $|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2}| \le \frac{2\sqrt{2}\,r}{1-r^2}$, $|z| = r < 1$;
3. **Distortion theorem**: for each $f \in \sigma$, we have $\frac{(1-r)^{\sqrt{2}-1}}{(1+r)^{\sqrt{2}+1}} \le |f'(z)| \le \frac{(1+r)^{\sqrt{2}-1}}{(1-r)^{\sqrt{2}+1}}$, $|z| = r < 1$;
4. **Rotation theorem**: for each $f \in \sigma$, we have $|\arg f'(z)| \le \sqrt{2}\log\frac{1+r}{1-r}$, $|z| = r < 1$;
5. **Growth theorem**: for each $f \in \sigma$, we have

$$\frac{1}{2\sqrt{2}}\left[1 - \left(\frac{1-r}{1+r}\right)^{\sqrt{2}}\right] \le \left|f(z)\right| \le \frac{1}{2\sqrt{2}}\left[\left(\frac{1+r}{1-r}\right)^{\sqrt{2}} - 1\right], \quad |z| = r < 1;$$

6. **Combined growth and distortion theorem**: for each $f \in \sigma$, we have

$$\frac{2\sqrt{2}\,r}{\frac{(1+r)^{\sqrt{2}+1}}{(1-r)^{\sqrt{2}-1}} - 1 + r^2} \le \left|\frac{zf'(z)}{f(z)}\right| \le \frac{2\sqrt{2}\,r}{1 - r^2 - \frac{(1-r)^{\sqrt{2}+1}}{(1+r)^{\sqrt{2}-1}}}, \quad |z| = r < 1.$$

7. **Radius of convexity**: For every positive number $\rho \le \sqrt{2} - 1$, each function $f \in \sigma$ maps the disk $|z| < \rho$ onto a convex domain.

Note that the problem of finding sharp estimates for all our theorem is still eluding us.

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