



ELSEVIER

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com



Partial differential equations

A note on the existence of extension operators for Sobolev spaces on periodic domains



Une note sur l'existence d'opérateurs d'extension pour les espaces de Sobolev sur des domaines périodiques

Martin Höpker, Michael Böhm

Center for Industrial Mathematics, FB 3, University of Bremen, Postfach 33 04 40, 28334 Bremen, Germany

ARTICLE INFO

Article history:

Received 13 November 2013

Accepted 3 September 2014

Available online 23 September 2014

Presented by Olivier Pironneau

ABSTRACT

In this note, we prove the existence of a family of extension operators for Sobolev spaces defined on ε -periodic domains. The norms of the operators are shown to be independent of ε . This extension theorem is relevant in the theory of homogenization for PDE's under flux boundary conditions.

© 2014 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

R É S U M É

Dans cette note, nous prouvons l'existence d'une famille d'opérateurs d'extension pour les espaces de Sobolev définis sur des domaines ε -périodiques. Nous montrons que les normes de ces opérateurs sont indépendantes de ε . Ce théorème est pertinent dans la théorie de l'homogénéisation des EDP avec des conditions aux limites de flux.

© 2014 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

This note is motivated by homogenizing settings involving PDE's under flux boundary conditions.

Homogenization of periodic porous media often leads to the question of knowing whether solutions to PDE's defined on the pore space, respectively on the solid matrix, can be extended to the whole medium in a way that preserves bounds independently of the size of the microstructure.

Formally, we consider an ε -periodic set Ω_ε (which may describe either the pore part of the porous medium or its solid matrix): let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, E a 1-periodic Lipschitz domain, $\varepsilon > 0$ and define $\Omega_\varepsilon := \Omega \cap (\varepsilon E)$. The above question then reads as: is there a bounded family of linear extension operators $L^\varepsilon : W^{1,p}(\Omega_\varepsilon) \rightarrow W^{1,p}(\Omega)$?

Such a family can quite easily be constructed if Ω “has isolated holes”, i.e. if the complement of Ω_ε is disconnected, basically by gluing together local extensions on adjacent cubes around those holes (see, e.g., [4]). In some cases, this even allows us to preserve bounds of derivatives (see [7]). Homogenization in this setting has been studied extensively, see e.g., among many others, [8,6].

E-mail addresses: hoepker@math.uni-bremen.de (M. Höpker), mbohm@math.uni-bremen.de (M. Böhm).

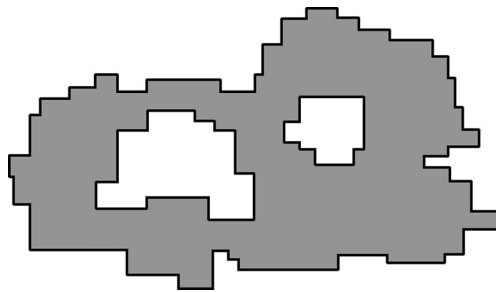


Fig. 1. Example of a domain Ω . / Exemple de domaine Ω .

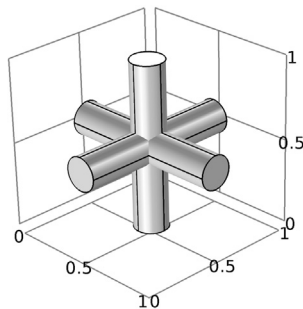


Fig. 2. Reference cell in the pipe model. / Cellule de référence pour le modèle de tuyau.

But if the complement of Ω_ε is connected, the approach to construct the extension operators mentioned above is not possible. The probably most well-known paper that tackles this problem is the classical article [1], in which the existence of extension operators $L^\varepsilon : W^{1,p}(\Omega_\varepsilon) \rightarrow W^{1,p}_{loc}(\Omega)$ is shown. But as, in general, it is impossible to control the behavior of the periodic microstructure at the boundary of Ω , norm estimates are in this case just valid at a distance of the boundary of Ω .

For functions having zero trace at the external boundary $\partial\Omega \cap \partial\Omega_\varepsilon$, the problem simplifies and can be dealt with by the result taken from [1] (see e.g. [5]), but a general result applicable to flux boundary conditions is not known. Some work has been done concerning similar extension problems for other classes of functions, e.g., SBV-functions in [3].

To avoid the problems at the external boundary, we restrict ourselves to a situation where we have full information on the microstructure near the boundary, which will allow us to modify the approach of [1] and prove the desired result.

We assume Ω to be representable by a finite union of axis-parallel cuboids (see Fig. 1 for an example), each of which is assumed to have corner coordinates in \mathbb{Q}^n . Let $\varepsilon > 0$ be chosen such that the stretched domain $\varepsilon^{-1}\Omega$ can be represented by a finite union of axis-parallel cuboids with corner coordinates in \mathbb{Z}^n . Denote $Y := (0, 1)^n$. Assume that $E \cap Y$ is a Lipschitz domain and that $E \cap \bar{Y}$ is identical on all faces of Y .

Note that by such a choice of Ω we can approximate any Lipschitz domain as accurately as necessary for any practical purpose and that the assumptions on E cover the very important case of the “pipe-model” for porous media (see Fig. 2). For other models of porous media, we refer to [2].

2. Further preliminaries and construction of the extension operators

For any set $A \subset \mathbb{R}^n$ and any multiindex $\alpha \in \mathbb{Z}^n$, we denote by A^α the translation of A by α . As in [1], we choose a 1-periodic partition of unity of \mathbb{R}^n : Let $Q := (-1; 1)^n$. For all $\alpha \in \mathbb{Z}^n$, there exists a smooth function $\phi^\alpha \geq 0$ such that $\text{supp}(\phi^\alpha) \subset\subset Q^\alpha$ and $\sum_{\alpha \in \mathbb{Z}^n} \phi^\alpha(x) = 1$ for all $x \in \mathbb{R}^n$ with the uniform estimates $|\phi^\alpha|, |\nabla\phi^\alpha| \leq M$.

We call a bounded Lipschitz domain $P \subset \mathbb{R}^n$ an “admissible object” (AO) if $P \cap E$ is a Lipschitz domain as well. From the assumptions on Ω and E , it is quite obvious that (for sufficiently small ε) we can find a finite collection of AOs, denoted by $P_l \subset \mathbb{R}^n$ with $l = 1 \dots L$ (where L is independent of ε), such that

$$(\alpha \in \mathbb{Z}^n : Q^\alpha \cap \varepsilon^{-1}\Omega \neq \emptyset) \Rightarrow (\exists \bar{\alpha} \in \mathbb{Z}^n, l^\alpha \in \{1, \dots, L\} : P_{l^\alpha}^{\bar{\alpha}} \subset \varepsilon^{-1}\Omega \wedge (Q^\alpha \cap \varepsilon^{-1}\Omega) \subset P_{l^\alpha}^{\bar{\alpha}}).$$

For a visualization in two dimensions, see Fig. 3. This reflects that cuboids can locally only intersect in a finite number of ways. Obviously, all P_l can be chosen as unions of cuboids with corner coordinates in \mathbb{Z}^n .

We now state a local extension result:

Lemma 2.1. *Let P be an AO, then there exists a linear operator $\tau_P : W^{1,p}(P \cap E) \rightarrow W^{1,p}(P)$ such that, for every $u \in W^{1,p}(P \cap E)$, the extension property*



Fig. 3. (Color online.) Example of AO's and their respective translations into $\epsilon^{-1}\Omega$ / Exemple des AO et leurs traductions respectives dans $\epsilon^{-1}\Omega$.

$$\tau_P u = u \quad \text{a.e. in } P \cap E$$

and the estimates

$$\int_P |\tau_P u|^p dx \leq C \int_{P \cap E} |u|^p dx,$$

$$\int_P |\nabla(\tau_P u)|^p dx \leq C \int_{P \cap E} |\nabla u|^p dx$$

are satisfied, where the constant C only depends on n, p, E and P .

Proof. Apply Lemma 2.6. from [1] to $A := P \cap E$ and $\omega := P$. \square

By a simple translation argument, a similar extension operator with the same constant C exists on $P^\alpha \cap E$ for any multiindex $\alpha \in \mathbb{Z}^n$. For a function $u \in W^{1,p}(P^\alpha \cap E)$, we denote its extension to P^α by u_l^α .

We observe that (again for ϵ small enough) $\epsilon^{-1}\Omega = \bigcup_{\beta \in I_{\epsilon^{-1}\Omega}} Q^\beta$, where $I_{\epsilon^{-1}\Omega} = \{\beta \in \mathbb{Z}^n : Q^\beta \subset \epsilon^{-1}\Omega\}$.

We state our main result:

Theorem 2.2. Under the assumptions above, there exists a family of linear operators $L^\epsilon : W^{1,p}(\Omega_\epsilon) \rightarrow W^{1,p}(\Omega)$ such that for every $u_\epsilon \in W^{1,p}(\Omega_\epsilon)$

$$L^\epsilon(u_\epsilon) = u_\epsilon \quad \text{on } \Omega_\epsilon,$$

$$\|L^\epsilon(u_\epsilon)\|_{W^{1,p}(\Omega)} \leq C \|u_\epsilon\|_{W^{1,p}(\Omega_\epsilon)}$$

where the constant $C > 0$ does not depend on ϵ .

Proof. In the following, any positive constant that is independent of ϵ will be denoted by C .

At first, we construct a linear extension operator $L^\epsilon : W^{1,p}(\epsilon^{-1}\Omega \cap E) \rightarrow W^{1,p}(\epsilon^{-1}\Omega)$ by

$$L^\epsilon(u) := \sum_{\alpha \in \mathbb{Z}^n : Q^\alpha \cap \epsilon^{-1}\Omega \neq \emptyset} u_{l^\alpha}^\alpha \phi_{|\epsilon^{-1}\Omega}^\alpha,$$

where the extension property is obvious and linearity results from the linearity of the local extension operators.

Let $\beta \in I_{\epsilon^{-1}\Omega}$. Define the set $I^\beta := \{\alpha \in \mathbb{Z}^n : Q^\alpha \cap Q^\beta \neq \emptyset\}$, the size of which is bounded independently of β and of ϵ .

For any $\alpha \in I^\beta$, we also have $\emptyset \neq Q^\alpha \cap Q^\beta \subset Q^\alpha \cap \epsilon^{-1}\Omega \subset P_{l^\alpha}^\alpha$, which implies in particular $P_{l^\alpha}^\alpha \cap Q^\beta \neq \emptyset$.

We estimate:

$$\int_{Q^\beta} |L^\epsilon(u)|^p dx = \int_{Q^\beta} \left| \sum_{\alpha \in \mathbb{Z}^n : Q^\alpha \cap \epsilon^{-1}\Omega \neq \emptyset} u_{l^\alpha}^\alpha \phi_{|\epsilon^{-1}\Omega}^\alpha \right|^p dx = \int_{Q^\beta} \left| \sum_{\alpha \in I^\beta} u_{l^\alpha}^\alpha \phi_{|\epsilon^{-1}\Omega}^\alpha \right|^p dx$$

$$\leq C \sum_{\alpha \in I^\beta} \int_{Q^\beta} |u_{l^\alpha}^\alpha \phi_{|\epsilon^{-1}\Omega}^\alpha|^p dx = C \sum_{\alpha \in I^\beta} \int_{Q^\beta \cap Q^\alpha} |u_{l^\alpha}^\alpha \phi_{|\epsilon^{-1}\Omega}^\alpha|^p dx \leq C \sum_{\alpha \in I^\beta} \int_{P_{l^\alpha}^\alpha} |u_{l^\alpha}^\alpha|^p dx,$$

where in the last inequality we used the boundedness of the ϕ^α .

We introduce the set $J^\beta := \{(l, \eta) \in \{1 \dots L\} \times \mathbb{Z}^n : P_l^\eta \cap Q^\beta \neq \emptyset, P_l^\eta \subset \epsilon^{-1}\Omega\}$, the size of which is bounded independently of β .

For any $\alpha \in I^\beta$, the application of the rough estimate:

$$\int_{P_{l^\alpha}^\alpha} |u_{l^\alpha}^\alpha|^p dx \leq \sum_{(l, \eta) \in J^\beta} \int_{P_l^\eta} |u_l^\eta|^p dx,$$

which is independent of α , and the subsequent application of the local extension theorems for each P_l^η yield:

$$\int_{Q^\beta} |L^\varepsilon(u)|^p dx \leq C \sum_{(l,\eta) \in J^\beta} \int_{P_l^\eta} |u_l^\eta|^p dx \leq C \sum_{(l,\eta) \in J^\beta} \int_{P_l^\eta \cap E} |u|^p dx.$$

Obviously, there exists $K > 0$, independently of β , such that $P_l^\eta \subset (KQ)^\beta$ for all $(l, \eta) \in J^\beta$. Hence

$$\int_{Q^\beta} |L^\varepsilon(u)|^p dx \leq C \int_{(KQ)^\beta \cap \varepsilon^{-1}\Omega \cap E} |u|^p dx.$$

We deduce:

$$\int_{\varepsilon^{-1}\Omega} |L^\varepsilon(u)|^p dx \leq \sum_{\beta \in I_{\varepsilon^{-1}\Omega}} \int_{Q^\beta} |L^\varepsilon(u)|^p dx \leq C \sum_{\beta \in I_{\varepsilon^{-1}\Omega}} \int_{(KQ)^\beta \cap \varepsilon^{-1}\Omega \cap E} |u|^p dx \leq C \int_{\varepsilon^{-1}\Omega \cap E} |u|^p dx,$$

where in the last inequality we used the fact that there exists a number $D \in \mathbb{N}$ such that any $x \in \mathbb{R}^n$ lies in at most D cubes of the form $(KQ)^\beta$ with $\beta \in \mathbb{Z}^n$.

We calculate:

$$\begin{aligned} \int_{Q^\beta} |\nabla(L^\varepsilon(u))|^p dx &= \int_{Q^\beta} \left| \sum_{\alpha \in \mathbb{Z}^n: Q^\alpha \cap \varepsilon^{-1}\Omega \neq \emptyset} (\nabla(u_{|\alpha}^{\bar{\alpha}})\phi_{|\varepsilon^{-1}\Omega}^\alpha + u_{|\alpha}^{\bar{\alpha}} \nabla\phi_{|\varepsilon^{-1}\Omega}^\alpha) \right|^p dx \\ &\leq 2^{p-1} \int_{Q^\beta} \left| \sum_{\alpha \in \mathbb{Z}^n: Q^\alpha \cap \varepsilon^{-1}\Omega \neq \emptyset} \nabla(u_{|\alpha}^{\bar{\alpha}})\phi_{|\varepsilon^{-1}\Omega}^\alpha \right|^p dx + 2^{p-1} \int_{Q^\beta} \left| \sum_{\alpha \in \mathbb{Z}^n: Q^\alpha \cap \varepsilon^{-1}\Omega \neq \emptyset} u_{|\alpha}^{\bar{\alpha}} \nabla\phi_{|\varepsilon^{-1}\Omega}^\alpha \right|^p dx. \end{aligned}$$

By proceeding similarly as above we can deduce:

$$\int_{\varepsilon^{-1}\Omega} |\nabla(L^\varepsilon(u))|^p dx \leq C \int_{\varepsilon^{-1}\Omega \cap E} |\nabla u|^p dx + C \int_{\varepsilon^{-1}\Omega \cap E} |u|^p dx.$$

The claimed existence of a linear extension operator $\bar{L}^\varepsilon : W^{1,p}(\Omega_\varepsilon) \rightarrow W^{1,p}(\Omega)$, whose norm is independent of ε , follows by a simple scaling argument, analogous to that in [1]. \square

References

- [1] E. Acerbi, V. Chiadò Piat, G. Dal Maso, D. Percivale, An extension theorem from connected sets, and homogenization in general periodic domains, *Nonlinear Anal.* 18 (1992) 481–496.
- [2] J. Bear, Y. Bachmat, *Introduction to Modeling of Transport Phenomena in Porous Media*, Kluwer Academic Publishers, 1990.
- [3] F. Cagnetti, L. Scardia, An extension theorem in SBV and an application to the homogenization of the Mumford–Shah functional in perforated domains, *J. Math. Pures Appl.* 95 (2011) 349–381.
- [4] D. Cioranescu, J. Saint, Jean Paulin, *Homogenization of Reticulated Structures*, Applied Mathematical Sciences, vol. 136, Springer, 1999.
- [5] M. Mabrouk, S. Hassan, Homogenization of a composite medium with a thermal barrier, *Math. Methods Appl. Sci.* 27 (2004) 405–425.
- [6] H.S. Mahato, *Homogenization of a system of nonlinear multi-species diffusion–reaction equations in an $H^{1,p}$ setting*, Ph.D. thesis, University of Bremen, Germany, 2013.
- [7] R. Miller, Extension theorems for homogenization on lattice structures, *Appl. Math. Lett.* 5 (1992) 73–78.
- [8] C. Timofte, Homogenization results for enzyme catalyzed reactions through porous media, *Acta Math. Sci. Ser. B* 29 (2009) 74–82.