C. R. Acad. Sci. Paris, Ser. I 352 (2014) 797–801

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Potential theory/Probability theory

Survival time of a heterogeneous random walk in a quadrant

Marches aléatoires dans un milieu hétérogène, temps de survie dans un quadrant

Néjib Ben Salem [a], Sami Mustapha [b], Mohamed Sifi [a]

[a] *Université de Tunis El Manar, Faculté des Sciences de Tunis, LR11ES11 Laboratoire d'analyse mathématiques et applications, 2092 Tunis, Tunisia*

[b] *Centre de mathématiques de Jussieu, Université Pierre-et-Marie-Curie (Paris-6), tour 46, 5e étage, boîte 247, 4, place Jussieu, 75252 Paris cedex 05, France*

A R T I C L E   I N F O

Article history:
Received 25 April 2014
Accepted after revision 26 August 2014
Available online 23 September 2014

Presented by the Editorial Board

A B S T R A C T

We obtain upper Gaussian estimates of transition probabilities of inhomogeneous random walks on the positive quadrant. Among the most important steps in our proof are comparison arguments based on discrete variants of the Harnack principle and large deviations estimates.

R É S U M É

Nous obtenons une estimation gaussienne supérieure des probabilités de transition d'une marche aléatoire hétérogène dans le quadrant positif. Les ingrédients essentiels de notre preuve sont des arguments de comparaison basés sur des variantes discrètes du principe de Harnack et des estimations du type grandes déviations.

Version française abrégée

On se place dans $\mathbb{Z}^2$ (la généralisation à $\mathbb{Z}^d$ est immédiate) et on considère la chaîne de Markov $(S_j)_{j\in\mathbb{N}}$ définie par

$$\mathbb{P}\big[S_{j+1}=\big(k',l'\big)\mid S_j=(k,l)\big]=\pi\big((k,l);\big(k',l'\big)\big),\quad (k,l);\big(k',l'\big)\in\mathbb{Z}^2,$$

$$\text{où } \pi:\mathbb{Z}^2\times\mathbb{Z}^2\to[0,1] \text{ est définie par } \begin{cases}\pi((k,l);(k\pm1,l))=\alpha_k\\ \pi((k,l);(k,l\pm1))=\beta_l\\ \pi((k,l);(k,l))=1-2(\alpha_k+\beta_l)\end{cases} \quad \text{où } (k,l)\in\mathbb{Z}^2$$

avec $\pi((k,l);(k',l'))=0$ pour toute autre valeur de (k',l').

E-mail addresses: nejib.bensalem@fst.rnu.tn (N. Ben Salem), sam@math.jussieu.fr (S. Mustapha), mohamed.sifi@fst.rnu.tn (M. Sifi).

http://dx.doi.org/10.1016/j.crma.2014.08.016
1631-073X/

On supposera que les probabilités $(\alpha_k)_{k\in\mathbb{Z}}$ et $(\beta_l)_{l\in\mathbb{Z}}$ vérifient la condition d'ellipticité suivante :

$$\alpha_k \geq \varepsilon, \quad \beta_l \geq \varepsilon, \quad \alpha_k + \beta_l \leq \frac{1-\varepsilon}{2}; \quad k, l \in \mathbb{Z} \tag{1}$$

pour une constante $\varepsilon > 0$.

Considérons maintenant $(\mathbb{Z}_+)^2 = \{(k,l) \in \mathbb{Z}^2 \mid k > 0, l > 0\}$ le quadrant positif de $\mathbb{Z}^2$ avec son bord $\partial(\mathbb{Z}_+)^2 = \{(k,0), k \geq 1\} \cup \{(0,l), l \geq 1\}$ et désignons par

- τ le temps de sortie de la marche de $(\mathbb{Z}_+)^2$: $\tau = \inf\{j = 0, 1, \cdots \mid S_j \in \partial(\mathbb{Z}_+)^2\}$,
- m la fonction de $\mathbb{Z}^2 \to \mathbb{R}_+$ définie par $m(k,l) = \frac{\alpha_0\beta_0}{\alpha_k\beta_l}$,
- V la fonction volume définie par $V((k,l); r) = \sum_{(k',l') \in B_r(k,l)} m(k', l')$, où $B_r(k,l)$ désigne l'ensemble des points à coordonnées entières de la boule euclidienne centrée en $(k,l) \in \mathbb{Z}^2$ et de rayon $r > 0$,
- $q_n((k,l); (k',l'))$ le noyau de transition de la marche tuée au bord du quadrant

$$q_n\big((k,l); (k',l')\big) = \mathbb{P}_{(k,l)}\big[S_n = (k',l'); \tau > n\big], \quad n = 1, 2, \cdots; \ (k,l), (k',l') \in (\mathbb{Z}_+)^2.$$

Notre résultat principal est le suivant.

Théorème 0.1. *Soient $(S_j)_{j\in\mathbb{N}}$ comme ci-dessus vérifiant la condition (1). Alors il existe $C_\varepsilon > 0$ (dépendant uniquement de ε) telle que pour tout $n \geq 1$ et pour tous $(k,l); (k',l') \in (\mathbb{Z}_+)^2$:*

$$q_n\big((k,l); (k',l')\big) \leq \frac{C_\varepsilon k k' l l'}{n^2 \alpha_{k'} \beta_{l'} V((k,l); \sqrt{n})} \exp\left(-\frac{(k-k')^2 + (l-l')^2}{C_\varepsilon n}\right). \tag{2}$$

Ce type d'estimation permet de déduire facilement une estimation supérieure du noyau de Green du quadrant pour cette marche.

Cette majoration a été obtenue par le deuxième auteur pour les marches aléatoires hétérogènes sur le demi-espace $\mathbb{Z}_+^d = \{x = (x', x_d) \in \mathbb{Z}^{d-1} \times \mathbb{Z}, x_d > 0\}$ (voir [10]).

Les auteurs conjecturent que l'estimation (2) reste vraie sous une hypothèse de contrôle plus faible de la manière dont les probabilités (α_k) et (β_l) s'approchent de 0. Une généralisation de (2) dans cette direction ouvrirait des perspectives intéressantes pour l'étude des propriétés de transience et de récurrence en milieu hétérogène.

La preuve repose essentiellement sur des arguments de comparaison et des estimations de type « grandes déviations » permettant de contrôler la probabilité de fuite de la marche dans le milieu hétérogène.

Les détails seront publiés ultérieurement.

1. Introduction

Random walks conditioned on staying in orthants of $\mathbb{Z}^d$ arise a great interest in the mathematical community, as they appear in several distinct domains: probability theory, enumerative combinatorics (cf. [1–5,13,14]), etc.

The aim of this work is to study spatially nonhomogeneous random walks on the quarter plane. Homogeneous random walks have been extensively studied by many authors via a systematic use of generating functions (cf. [1,4,5,13], etc.). Such a tool is not of great help in the inhomogeneous case (cf. [6,8–10] and [15]) and alternative methods should be used. The proof of the main result in this note is based on potential theory tools and large deviations inequalities.

Let us consider the Markov chain $(S_j)_{j\in\mathbb{N}}$ defined by

$$\mathbb{P}\big[S_{j+1} = (k',l') \mid S_j = (k,l)\big] = \pi\big((k,l); (k',l')\big), \quad (k,l); (k',l') \in \mathbb{Z}^2,$$

$$\text{with } \pi : \mathbb{Z}^2 \times \mathbb{Z}^2 \to [0,1] \text{ given by } \begin{cases} \pi((k,l); (k\pm 1, l)) = \alpha_k \\ \pi((k,l); (k, l\pm 1)) = \beta_l \\ \pi((k,l); (k,l)) = 1 - 2(\alpha_k + \beta_l) \end{cases} \quad \text{where } (k,l) \in \mathbb{Z}^2$$

and $\pi((k,l); (k',l')) = 0$ for all any other value (k',l').

$(S_j)_{j\in\mathbb{N}}$ is a spatially heterogeneous random walk with bounded symmetric increments that can visit at each step five potential sites, moving left or right with probability α_k, up or down with probability β_l, or remaining in place with probability $1 - 2(\alpha_k + \beta_l)$. The spatial heterogeneity results in a k-dependence (resp. l-dependence) of the probabilities transition when the displacement is parallel to the k-axis (resp. l-axis).

We suppose that the probabilities $(\alpha_k)_{k\in\mathbb{Z}}$ and $(\beta_k)_{k\in\mathbb{Z}}$ satisfy the following ellipticity condition:

$$\alpha_k \geq \varepsilon, \quad \beta_l \geq \varepsilon, \quad \alpha_k + \beta_l \leq \frac{1-\varepsilon}{2}; \quad k, l \in \mathbb{Z} \tag{3}$$

for some constant $\varepsilon > 0$.

The random walk $(S_j)_{j\in\mathbb{N}}$ has the advantage of not being a random walk with a Cartesian product structure and can help to model the general case of inhomogeneous random walk in $\mathbb{Z}^2$ while deviating little from the simple random walk model. It also offers the possibility to compute exact expressions for natural quantities controlling the transition probabilities kernel.

Consider $(\mathbb{Z}_+)^2 = \{(k,l) \in \mathbb{Z}^2 \mid k > 0, l > 0\}$ the positive quadrant of $\mathbb{Z}^2$ with its boundary $\partial(\mathbb{Z}_+)^2 = \{(k,0), k \geq 1\} \cup \{(0,l), l \geq 1\}$ and denote by

- τ the exit time of the walk $(\mathbb{Z}_+)^2$: $\tau = \inf\{j = 0, 1, \cdots \mid S_j \in \partial(\mathbb{Z}_+)^2\}$,
- m the function $\mathbb{Z}^2 \to \mathbb{R}_+$ defined by $m(k,l) = \frac{\alpha_0\beta_0}{\alpha_k\beta_l}$,
- V the volume function defined by $V((k,l);r) = \sum_{(k',l')\in B_r(k,l)} m(k',l')$, where $B_r(k,l)$ denotes the "discrete" Euclidean ball centered at $(k,l) \in \mathbb{Z}^2$ with radius $r > 0$,
- $q_n((k,l);(k',l'))$ the transition kernel corresponding to the walk killed at the boundary of the quadrant $q_n((k,l);(k',l')) = \mathbb{P}_{(k,l)}[S_n = (k',l'); \tau > n]$, $n = 1, 2, \cdots$; $(k,l), (k',l') \in (\mathbb{Z}_+)^2$.

Precise estimates of the kernel $q_n((k,l);(k',l'))$ can be given in terms of the function m defining the above volume function V and a geometric factor that controls the behavior of the walk as it approaches the boundary. Our main result is the following:

Theorem 1.1. *Let $(S_j)_{j\in\mathbb{N}}$ as above satisfying the condition (3). Then there exists $C_\varepsilon > 0$, depending uniquely only on ε, such that for all $n \geq 1$; and (k,l); $(k',l') \in (\mathbb{Z}_+)^2$*

$$q_n\big((k,l);\big(k',l'\big)\big) \leq \frac{C_\varepsilon kk'll'}{\alpha_{k'}\beta_{l'}V((k,l);\sqrt{n})n^2}\exp\bigg(-\frac{(k-k')^2+(l-l')^2}{C_\varepsilon n}\bigg).$$

The following comments may be helpful in placing the above theorem in its proper perspective.

(i) It is worth mentioning that the constant C_ϵ depends only on the ellipticity constant ϵ, but in a very complicated way. This is due to the use in the proof of various Harnack inequalities and the non-obvious dependence of the constants involved in these inequalities on ϵ.

(ii) In the case of a homogeneous random walk, the function $m \equiv 1$ and the volume induced by m is the standard one.

(iii) It is clear that, under the ellipticity condition, the volume function V satisfies $V((k,l);r) \approx r^2$ and the main estimate can be rewritten as

$$q_n\big((k,l);\big(k',l'\big)\big) \leq \frac{C'_\varepsilon kk'll'}{n^3}\exp\bigg(-\frac{(k-k')^2+(l-l')^2}{C_\varepsilon n}\bigg). \tag{4}$$

Summing over n varying from 1 to infinity and using a dyadic decomposition then comparing with continuous integrals, we obtain the following upper bound of the Green kernel of the quadrant for the random walk:

$$G\big((k,l);\big(k',l'\big)\big) \leq C_\varepsilon \frac{kk'll'}{((k-k')^2+(l-l')^2)^2}.$$

This result can be compared with the Green functions estimates obtained by Raschel [11] in the homogeneous case.

(iv) In the same way, summing over $(k',l') \in (\mathbb{Z}_+)^2$, one can also deduce from the estimate (4) that the survival time of the random walk in the quadrant satisfies:

$$\mathbb{P}_{(k,l)}[\tau > n] \leq C_\varepsilon \frac{kl}{n}.$$

(v) In the case of an orthant $(\mathbb{Z}_+)^d$, the estimate (4) generalizes as follows:

$$q_n\big((x_1,\cdots,x_d);(y_1,\cdots,y_d)\big) \leq \frac{C_\varepsilon x_1.y_1\cdots x_d.y_d}{n^{3d/2}}\exp\bigg(-\frac{|x-y|^2}{C_\varepsilon n}\bigg), \tag{5}$$

with obvious notation.

2. Sketch of the proof

In what follows, we give an overview of the proof of the main estimate. This proof is based on a systematic use of potential theory tools and large deviations inequalities. We follow the approach adopted in the case of a half-space (cf. [10]).

We first establish analogues of the discrete parabolic and boundary Harnack principles obtained in the case of half-space (cf. [10]). In fact these results are valid for general spatially inhomogeneous random walks on $\mathbb{Z}^d$ ($d \geq 2$), having bounded symmetric increments and satisfying an ellipticity condition.

The large deviation estimate used to control the mass escape for $(S_j)_{j\in\mathbb{N}}$ is a generalization of a well-known estimate for a simple random walk (see [7]).

Theorem 2.1. *For all $a>0$, there exists a positive constant C_a such that for all $n,t>0$ and $(k,l)\in\mathbb{Z}^2$*

$$\mathbb{P}_{(k,l)}\big[\big|S_n-(k,l)\big|>at\sqrt{n}\big]\le C_a\exp\big(-t^2\big).$$

We note that the ellipticity condition is not necessary for the proof of Theorem 2.1. We also need a variant of Theorem 2.1, which allows us to control the mass escape at the boundary.

Theorem 2.2. *For all $a>0$, there exists a positive constant $C_{a,\varepsilon}$ such that for all $n,t>0$ and $(k,l)\in(\mathbb{Z}_+)^2$*

$$\mathbb{P}_{(k,l)}\big[\big|S_n-(k,l)\big|>at\sqrt{n},\ \tau>n\big]\le C_{a,\varepsilon}\frac{kl}{n}\exp\big(-t^2\big).$$

The proof of Theorem 2.2 is based on the boundary Harnack principle and requires the use of the ellipticity condition.

Around the functions m and $u:(\mathbb{Z}_+)^2\to\mathbb{R}_+$, $(k,l)\mapsto kl$. Let us introduce the generator corresponding to the chain (S_j), that is the difference operator L acting on functions $f:\mathbb{Z}^2\to\mathbb{R}$ defined by:

$$Lf(k,l)=\alpha_k\big(f(k+1,l)+f(k-1,l)\big)+\beta_l\big(f(k,l+1)+f(k,l-1)\big)-2(\alpha_k+\beta_l)f(k,l),$$

and its adjoint:

$$L^*g(k,l)=\alpha_{k+1}g(k+1,l)+\beta_{l+1}g(k,l+1)+\alpha_{k-1}g(k-1,l)+\beta_{l-1}g(k,l-1)-2(\alpha_k+\beta_l)g(k,l).$$

An immediate computation shows that $L^*m(k,l)=0$, $(k,l)\in\mathbb{Z}^2$ and $Lu(k,l)=0$, $(k,l)\in(\mathbb{Z}_+)^2$.

These two functions that appear in the main estimate satisfy also the following fundamental uniqueness property.

Lemma 2.3. *The function m is the unique positive function (up to a multiplicative constant) satisfying*

$$L^*m(k,l)=0,\quad (k,l)\in\mathbb{Z}^2.$$

The previous lemma is an immediate consequence of [9, Theorem 8].

Lemma 2.4. *The function u is the unique positive function (up to a multiplicative constant) satisfying*

$$Lu(k,l)=0,\quad (k,l)\in(\mathbb{Z}_+)^2;\qquad u|_{\partial(\mathbb{Z}_+)^2}=0.$$

This lemma is valid as soon as the random walk satisfies symmetry and ellipticity conditions. Its proof follows essentially the same line as that of Theorem 9 of [10] and is based on the elliptic version of the boundary Harnack principle.

Before giving an idea on the proof of Theorem 1.1, we need to introduce a final ingredient that will play a key role in the proof, that is the notion of caloric function.

Definition 2.1. A function $v:\overline{(\mathbb{Z}_+)^2}\times\mathbb{N}\to\mathbb{R}$ is said caloric on $(\mathbb{Z}_+)^2\times\mathbb{N}^\star$ if

$$v\big((k,l);n+1\big)-v\big((k,l);n\big)=\sum_{(k',l')\in\mathbb{Z}^2}\pi\big((k,l);(k',l')\big)\big(v\big((k',l');n\big)-v\big((k,l);n\big)\big),\quad \big((k,l);n\big)\in(\mathbb{Z}_+)^2\times\mathbb{N}^\star.$$

The concept of caloric function is a space–time generalization of the notion of harmonic function. Caloric functions share with harmonic functions their fundamental properties: maximum principle, Harnack principle, and boundary Harnack principle. The techniques of potential theory therefore apply very well to caloric functions. The main difference lies in the fact that when using Harnack inequalities, we must let the time flow. We observe that for fixed $(k',l')\in(\mathbb{Z}_+)^2$, the function $((k,l);n)\to q_n((k,l);(k',l'))$ is caloric on $(\mathbb{Z}_+)^2\times\mathbb{N}^\star$.

Idea on the proof of Theorem 1.1. Three cases should be distinguished:

Case I. Both points (k,l) and (k',l') are at a distance $\ge\sqrt{n}$ away from the boundary. We consider the function $((k',l');n)\to\alpha_{k'}\beta_{l'}p_n((k,l);(k',l'))/\alpha_k\beta_l$ where (k,l) is fixed and $p_n((k,l);(k',l'))$ denotes the global transition kernel of the walk. An immediate computation shows that this function is caloric. Applying the parabolic Harnack principle, combining with Theorem 2.1 and using the obvious inequality $p_n((k,l);(k',l'))\ge q_n((k,l);(k',l'))$, we deduce the main estimate in this case.

Case II. One of the two points is close to the boundary and the other is far, say $\mathrm{dist}((k',l');\partial(\mathbb{Z}_+)^2) \geq \sqrt{n}$ and $\mathrm{dist}((k,l);\partial(\mathbb{Z}_+)^2) \leq \sqrt{n}$. In this case, the boundary Harnack principle allows us to compare the two positive caloric functions $q_n(.;(k',l'))$ and $u(.)$ near the boundary, and then the large deviation estimate of Theorem 2.2 is used essentially to bring up the Gaussian factor.

Case III. Both points (k,l) and (k',l') are at a distance $\leq \sqrt{n}$ from the boundary. We consider the caloric function $((k',l');n) \to \alpha_{k'}\beta_{l'}q_n((k,l);(k',l'))/\alpha_k\beta_l$ vanishing on the boundary and use the boundary Harnack principle to compare it with the function $u(.)$ and Theorem 2.2 to incorporate the Gaussian factor. □

Remarks.

(i) The exponent $\frac{3d}{2}$ which appears in (5) can be decomposed as $\frac{3d}{2} = \frac{d}{2} + d$. The term $n^{d/2}$ corresponds to the volume of the ball of radius $\sqrt{n}$. The term n^d fits very well with the homogeneity of the function $u : (\mathbb{Z}_+)^d \to \mathbb{R}_+$, $x = (x_1, \ldots, x_d) \mapsto x_1 \ldots x_d$ since we can rewrite the quotient

$$\frac{x_1.y_1 \ldots x_d.y_d}{n^d} = u\left(\frac{x_1}{\sqrt{n}}, \ldots, \frac{x_d}{\sqrt{n}}\right)u\left(\frac{y_1}{\sqrt{n}}, \ldots, \frac{y_d}{\sqrt{n}}\right)$$

for points with suitable coordinates and for adequate times. This reflects the fact that, despite the heterogeneity of the environment, the global space–time homogeneity still governs the behavior of the walk. This homogeneity is also reflected by the Gaussian factor and by the square root in the volume factor.

In contrast with this behavior, an interesting case to study is the one considered by Raschel (cf. [13, Figure 7, p. 20] and [12]). In this case, the analogue of our function u is not homogeneous.

(ii) The authors conjecture that the main estimate remains true under a much weaker hypothesis than the ellipticity condition (3), provided that a lower control on how the probabilities (α_k) and (β_k) approach 0 is imposed. On the other hand, the proof of the main estimate uses in an essential way the symmetry of the random walk (i.e. $\pi((k,l);(k+1,l)) = \pi((k,l);(k-1,l))$ and $\pi((k,l);(k,l+1)) = \pi((k,l);(k,l-1))$). It would be interesting to relax this assumption to include the case of walks with a drift in the spirit of the work [6].

References

[1] M. Bousquet-Mélou, Walks in the quarter plane: Krewera's algebraic model, Ann. Appl. Probab. 15 (2005) 1451–1491.
[2] M. Bousquet-Mélou, M. Mishna, Walks with small steps in the quarter plane, Contemp. Math. 520 (2010) 1–40.
[3] D. Denisov, V. Wachtel, Random walks in cones, Ann. Probab. (2014), in press.
[4] G. Fayolle, R. Iasnogorodski, V. Malyshev, Random Walks in the Quarter-Plane (Algebraic Methods, Boundary Value Problems and Applications), Appl. Math., vol. 40, Springer-Verlag, Berlin, 1999.
[5] I. Kurkova, K. Rashel, Random walks in the quarter plane, Bull. Soc. Math. Fr. 139 (2011) 341–387.
[6] P. Lafitte-Godillon, K. Raschel, V.C. Tran, Extinction probabilities for a distylous plant population modeled by an inhomogeneous random walk on the positive quadrant, SIAM J. Appl. Math. 73 (2013) 700–722.
[7] G.F. Lawler, Intersections of Random Walks, Birkhäuser/Springer, New York, 2013.
[8] I.M. MacPhee, M.V. Menshikov, A.R. Wade, Moments of exit times from wedges for non-homogeneous random walks with asymptotically zero drifts, J. Theor. Probab. 26 (1) (2013) 1–30.
[9] S. Mustapha, Gaussian estimates for stability inhomogeneous random walks on $\mathbb{Z}^d$, Ann. Probab. 34 (1) (2006) 264–283.
[10] S. Mustapha, Gambler's ruin estimates for random walks with symmetric spatially inhomogeneous increments, Bernoulli 13 (1) (2007) 131–147.
[11] K. Raschel, Green functions and Martin compactification for killed random walks related to SU(3), Electron. Commun. Probab. 15 (2010) 176–190.
[12] K. Raschel, Green functions for killed random walks in the Weyl chamber of Sp(4), Ann. Inst. Henri Poincaré Probab. Stat. 47 (2011) 1001–1019.
[13] K. Raschel, Random walks in the quarter plane, discrete harmonic functions and conformal mappings, with an appendix by Sandro Franceschi, Stoch. Process. Appl. 124 (2014) 3147–3178.
[14] N.Th. Varopoulos, Potential theory in conical domains, Math. Proc. Camb. Philos. Soc. 125 (1999) 335–384.
[15] N.Th. Varopoulos, Potential theory in conical domains II, Math. Proc. Camb. Philos. Soc. 129 (2000) 301–319.