C. R. Acad. Sci. Paris, Ser. I 352 (2014) 803–806

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Ordinary differential equations

The master Painlevé VI heat equation

L'équation maîtresse de la chaleur associée à Painlevé VI

Robert Conte [a,b,c], Ivan Dornic [b]

[a] *LRC MESO, Centre de mathématiques et de leurs applications (UMR 8536), et CEA-DAM, École normale supérieure de Cachan, 61, avenue du Président-Wilson, 94235 Cachan cedex, France*

[b] *Service de physique de l'état condensé (CNRS URA 2464), Orme des merisiers, CEA-Saclay, 91191 Gif-sur-Yvette cedex, France*

[c] *Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong*

ARTICLE INFO

Article history:
Received 26 June 2014
Accepted 9 August 2014
Available online 18 September 2014

Presented by Philippe G. Ciarlet

ABSTRACT

Given the second-order scalar Lax pair of the sixth Painlevé equation, we build a generalized heat equation with rational coefficients which does not depend any more on the Painlevé variable.

RÉSUMÉ

Étant donné la paire de Lax scalaire de la sixième équation de Painlevé, nous donnons une construction directe de l'équation de la chaleur généralisée à coefficients rationnels qui ne dépend plus de la variable de Painlevé.

Version française abrégée

Soit l'équation différentielle ordinaire (EDO) (1), dotée de quatre singularités fuchsiennes $x = x_\nu = \infty, 0, 1, t$ et d'une singularité apparente $t = u$. La condition d'isomonodromie (indépendance de la monodromie envers le birapport t des quatre singularités fuchsiennes) équivaut à une condition différentielle entre u et t, qui a ainsi conduit R. Fuchs [6] à la découverte de la sixième équation de Painlevé P6 (2).

Le processus d'isomonodromie conduit à adjoindre à l'EDO linéaire (1) une deuxième équation linéaire, ce couple (3), (4) définissant en langage moderne une paire de Lax scalaire.

Le but de cet article est de donner, pour la première fois, une preuve constructive de l'existence d'une équation de la chaleur généralisée, voir (20), dont les coefficients sont indépendants de la fonction de Painlevé u et ne dépendent, sous une forme rationnelle, que du birapport t et de la variable x. Les deux démonstrations antérieures n'étaient valides, comme détaillé dans la Section 3, que sous la condition $(\theta_\infty^2, \theta_0^2, \theta_1^2, \theta_t^2) \neq (1, 1, 1, 1)$, voir relation (10).

E-mail addresses: Robert.Conte@cea.fr (R. Conte), Ivan.Dornic@cea.fr (I. Dornic).

http://dx.doi.org/10.1016/j.crma.2014.08.006

1. Introduction. The scalar Lax pair of the sixth Painlevé equation

Let us first recall the 1905 classical result of R. Fuchs [6]. Consider a second-order linear ordinary differential equation (ODE) for a function $\psi=\psi(x)$ — the wave function, or wave vector — with four Fuchsian singularities of cross ratio t, put for convenience (but without loss of generality after a homographic transformation) at $x=x_\nu=\infty,0,1,t$. As prescribed by Poincaré to have sufficient degrees of freedom for the isomonodromy problem to be non-trivial [14, pp. 217–220], one must in addition put one apparent singularity located at $x=u$, so that the ODE satisfied by ψ writes [6, Eq. (1)]:

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}-\left[\frac{A}{x^2}+\frac{B}{(x-1)^2}+\frac{C}{(x-t)^2}+\frac{E}{(x-u)^2}+\frac{a}{x}+\frac{b}{x-1}+\frac{c}{x-t}+\frac{e}{x-u}\right]\psi=0. \tag{1}$$

In Eq. (1), A,B,C,E are constant parameters (independent of t and x), while a,b,c,e will ultimately depend on t but not on x. The requirement that the monodromy matrix (which transforms two independent solutions ψ_1,ψ_2 when x goes around a singularity x_ν) be independent of the location of the nonapparent singularity t — the isomonodromy condition — results in the constraint that u, as a function of the deformation parameter t, obeys the nonlinear second-order ordinary differential equation,

$$\begin{aligned}\frac{\mathrm{d}^2u}{\mathrm{d}t^2}&=\frac{1}{2}\left[\frac{1}{u}+\frac{1}{u-1}+\frac{1}{u-t}\right]\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2-\left[\frac{1}{t}+\frac{1}{t-1}+\frac{1}{u-t}\right]\frac{\mathrm{d}u}{\mathrm{d}t}\\&\quad+\frac{u(u-1)(u-t)}{t^2(t-1)^2}\left[\alpha+\beta\frac{t}{u^2}+\gamma\frac{t-1}{(u-1)^2}+\delta\frac{t(t-1)}{(u-t)^2}\right].\end{aligned} \tag{2}$$

Eq. (2) is the celebrated sixth Painlevé equation P6, the most general second-order nonlinear ODE without movable critical singularities, which lies at the crossroads of many problems of mathematics and theoretical physics of current active interest. The set of four parameters $(\alpha,\beta,\gamma,\delta)$ is in one-to-one correspondence with the set A,B,C,E defined in (1) and with the squares θ_ν^2 of the monodromy exponents, see relations (10) and (8) below.

Coming back to Eq. (1), what happens more precisely is that demanding the isomonodromy of Eq. (1) is tantamount to the existence of two linear equations for the wave vector (now a function $\psi=\psi(x,t)$). To endorse a modern terminology, the corresponding Fuchs–Garnier *scalar Lax pair* of equations writes [6–8]:

$$\partial_x^2\psi+(S/2)\psi=0, \tag{3}$$

$$\partial_t\psi+W\partial_x\psi-(1/2)W_x\psi=0, \tag{4}$$

and their commutativity (or compatibility) condition yields P6. In their most concise form (first exhibited by Garnier), the two scalar functions S,W display a remarkable symmetry between x and u:

$$-\frac{S}{2}=\frac{3/4}{(x-u)^2}+\frac{g_1u'+g_0}{(x-u)x(x-1)}+\frac{[(g_1u')^2-g_0^2]\frac{u-t}{u(u-1)}+f_G(u)}{x(x-1)(x-t)}+f_G(x), \tag{5}$$

$$W=-\frac{x(x-1)(u-t)}{(x-u)t(t-1)},\qquad g_1=-\frac{t(t-1)}{2(u-t)},\qquad g_0=-u+\frac{1}{2}, \tag{6}$$

$$f_G(z)=\frac{A}{z^2}+\frac{B}{(z-1)^2}+\frac{C}{(z-t)^2}+\frac{E}{z(z-1)}, \tag{7}$$

$$(2\alpha,-2\beta,2\gamma,1-2\delta)=\big(4(A+B+C+E+1),4A+1,4B+1,4C+1\big). \tag{8}$$

The purpose of this paper is to eliminate the dependent variable u (and its derivative) between the two linear equations (3), (4) while preserving the linearity of the resulting single equation. This provides us with a heat equation for the wave vector whose coefficients are solely rational functions of t and x (and of the monodromy parameters θ_ν). This generalized heat equation had in fact appeared earlier in the literature. We shall discuss this in the final section of the present Note. The paper is organized as follows. In Section 2, we present the elimination procedure. In Section 3, we compare our findings with previous results, discussing in particular the underlying motivation.

2. From the scalar Lax pair to the generalized heat equation

The guideline of our procedure is the singularity structure of the two linear equations (3), (4) in the complex plane of x. To achieve our goal, it is necessary (but not sufficient) to eliminate the polar singularity $x=u$ between the two equations.

The first equation (3) is an ODE with five Fuchsian singularities in the complex plane of x whose Riemann scheme is

$$\begin{pmatrix}\infty & 0 & 1 & t & u\\ (1-\theta_\infty)/2 & (1-\theta_0)/2 & (1-\theta_1)/2 & (1-\theta_t)/2 & -1/2\\ (1+\theta_\infty)/2 & (1+\theta_0)/2 & (1+\theta_1)/2 & (1+\theta_t)/2 & 3/2\end{pmatrix}, \tag{9}$$

with the correspondence

$$(2\alpha, -2\beta, 2\gamma, 1-2\delta) = \left(\theta_\infty^2, \theta_0^2, \theta_1^2, \theta_t^2\right). \tag{10}$$

As to the second equation (4), its ODE reduction $\partial_x = 0$ possesses one Fuchsian singularity at $x = u$, with the Riemann scheme

$$\begin{pmatrix} u \\ -1/2 \end{pmatrix}. \tag{11}$$

In a first step, we remove the four finite double poles in (3), *via* the change of wave function

$$\psi = x^{(1-\theta_0)/2}(x-1)^{(1-\theta_1)/2}(x-t)^{(1-\theta_t)/2}(x-u)^{-1/2}e^{G(t)}\Psi, \tag{12}$$

in which the gauge G is a function of t that is left for the moment arbitrary. The change of wave function (12) is quite similar to the classical one for the Gauss hypergeometric equation. After decomposition of its coefficients in simple elements of x, the Lax pair becomes

$$\begin{aligned}&\partial_x^2\Psi + \left(\frac{1-\theta_0}{x} + \frac{1-\theta_1}{x-1} + \frac{1-\theta_t}{x-t} - \frac{1}{x-u}\right)\partial_x\Psi \\ &\quad + \frac{1}{4u(u-1)(u-t)}\left(\frac{R_0}{x} + \frac{R_1}{x-1} + \frac{R_t}{x-t} + \frac{2R_u}{x-u}\right)\Psi = 0,\end{aligned} \tag{13}$$

$$t(t-1)\partial_t\Psi - \frac{x(x-1)(u-t)}{x-u}\partial_x\Psi + \left(t(t-1)G' + \frac{R_u}{2(x-u)} + \frac{(\theta_0+\theta_1+\theta_t-1)(u-t)}{2}\right)\Psi = 0, \tag{14}$$

in which the residues R_j are best expressed in terms of the two relations defining the one-parameter classical Riccati solution of P6 in terms of the hypergeometric function [7],

$$R(\theta_0, \theta_1, \theta_t) \equiv t(t-1)u' + u(u-1)(u-t)\left(\frac{\theta_0}{u} + \frac{\theta_1}{u-1} + \frac{\theta_t - 1}{u-t}\right) = 0, \tag{15}$$

$$\vartheta \equiv (1-\theta_0-\theta_1-\theta_t)^2 - \theta_\infty^2 = 0. \tag{16}$$

These residues are

$$\begin{aligned}R_u &= R(\theta_0, \theta_1, \theta_t),\\ R_0 &= -\frac{R(\theta_0, \theta_1, \theta_t)R(2-\theta_0, -\theta_1, -\theta_t) + \vartheta u(u-1)(u-t)u}{t},\\ R_1 &= -\frac{R(\theta_0, \theta_1, \theta_t)R(-\theta_0, 2-\theta_1, -\theta_t) + \vartheta u(u-1)(u-t)(u-1)}{(1-t)},\\ R_t &= -\frac{R(\theta_0, \theta_1, \theta_t)R(-\theta_0, -\theta_1, 2-\theta_t) + \vartheta u(u-1)(u-t)(u-t)}{t(t-1)}.\end{aligned} \tag{17}$$

It is remarkable that $-R_t/(4u(u-1)(u-t))$ is precisely the polynomial Hamiltonian of P6 [11].

In a second step, we eliminate this simple pole. Since the quotient of the residues of (13) and (14) at the simple pole $x = u$ does not depend on Ψ, the resulting equation remains linear in Ψ,

$$\begin{aligned}&-\frac{t(t-1)}{x(x-1)(x-t)}\partial_t\Psi + \partial_x^2\Psi - \left(\frac{\theta_0-1}{x} + \frac{\theta_1-1}{x-1} + \frac{\theta_t}{x-t}\right)\partial_x\Psi \\ &\quad + \frac{1}{x(x-1)(x-t)}\left[\frac{\vartheta}{4}(x-t) - t(t-1)G'(t) - F(t)\right]\Psi = 0,\end{aligned} \tag{18}$$

and its dependence on u and u' is gathered in an expression independent of x,

$$F(t) = -\frac{R(\theta_0, \theta_1, \theta_t)R(-\theta_0, -\theta_1, -\theta_t)}{4u(u-1)(u-t)} + \left(\theta_\infty^2 + 1 - (\theta_0+\theta_1+\theta_t)^2\right)\frac{u-t}{4}. \tag{19}$$

The third and last step is to choose the arbitrary function $G(t)$ so as to cancel this contribution of u and u'. The final result is a generalized heat equation whose coefficients are rational functions of t and x,

$$-t(t-1)\partial_t\Psi + x(x-1)(x-t)\left[\partial_x^2\Psi - \left(\frac{\theta_0-1}{x} + \frac{\theta_1-1}{x-1} + \frac{\theta_t}{x-t}\right)\partial_x\Psi\right] + \left[\frac{\vartheta}{4}(x-t) - g(t)\right]\Psi = 0, \tag{20}$$

in which $g(t)$ can be arbitrarily chosen. In the Picard case $\theta_\nu = 0$, its reduction $\partial_t = 0$, $g(t) = 0$ is identical to the classical linear ODE of Legendre for the periods of the elliptic function.

3. Discussion

The heat equation we have obtained is in fact not new, but the present proof is the first one without any restriction. Indeed, previous occurrences of this heat equation are the following.

(i) By establishing a formal correspondence between the scalar Lax pair (3), (4) and the time-dependent Schrödinger equation of quantum mechanics, Suleimanov [15] obtained this heat equation.
(ii) Starting from the second-order matrix Lax pair of P6 as given by Jimbo and Miwa [9], in which the monodromy matrix is just the sum of four simple poles, D.P. Novikov [12] assumed, like Ref. [9], that the residue at ∞ is a constant matrix and finally proved that the first component of the two-dimensional wave vector obeys the heat equation (20).
(iii) In a context of quantization of classical integrable systems (like Ref. [15]), Zabrodin and Zotov [17] also started from the matrix Lax pair of Ref. [9] and, after a suitable gauge transformation and change of variables, obtained a master P6 heat equation of the form

$$\partial_T \Psi = (1/2)\partial_X^2 \Psi + V(X,T)\Psi. \tag{21}$$

This time-dependent Schrödinger (or Fokker–Planck) equation coincides with the rational heat equation (20) after a point transformation $t \to T = T(t), x \to X = X(x,t)$ involving elliptic functions totally analogous to the transformation [6,13] that maps P6 to a Hamiltonian which is the sum of a kinetic energy and a time-dependent potential energy.

The drawback with the derivations of Refs. [12] and [17] is that, because of the assumption made in Ref. [9] that the residue at ∞ is a constant matrix, the matrix Lax pair as assumed by Jimbo and Miwa does not exist when all four θ_ν^2 are unity, see details in [10,3].

The earliest occurrence of the heat equation (20) which we are aware of is in conformal field theory [1, Eq. (5.17)] in the particular case of a value $c = 1$ of the central charge of a Virasoro algebra.

From the present results one deduces easily by the classical confluence of the four singularities [13,4] similar results for all the other Painlevé functions. In particular, the Tracy–Widom probability distribution in random matrix theory [16] has been recently characterized by such a time-dependent Schrödinger equation [2], associated with the second Painlevé function. Similar results hold for the sine-Gordon third Painlevé function [5].

References

[1] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, Nucl. Phys. B 241 (1984) 333–380.
[2] A. Bloemendal, B. Virág, Limits of spiked random matrices I, Probab. Theory Relat. Fields 156 (2013) 795–825, http://arxiv.org/abs/1011.1877.
[3] R. Conte, On the Lax pairs of the sixth Painlevé equation, RIMS Kokyuroku Bessatsu B 2 (2007) 21–27, http://arXiv.org/abs/nlin.SI/0701049.
[4] R. Conte, M. Musette, The Painlevé Handbook, Springer, Berlin, 2008; Metod Penleve y Ego Prilozhenia (Regular and Chaotic Dynamics), Springer, Berlin, 2011.
[5] I. Dornic, Phase-noise distribution, Brownian motion in time-dependent potentials, and the sine-Gordon Painlevé III transcendent, in preparation.
[6] R. Fuchs, Sur quelques équations différentielles linéaires du second ordre, C. R. Acad. Sci. Paris 141 (1905) 555–558.
[7] R. Fuchs, Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegenen wesentlich singulären Stellen, Math. Ann. 63 (1907) 301–321.
[8] R. Garnier, Sur des équations différentielles du troisième ordre dont l'intégrale générale est uniforme et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critiques fixes, Ann. Sci. Éc. Norm. Super. 29 (1912) 1–126.
[9] M. Jimbo, T. Miwa, Monodromy preserving deformations of linear ordinary differential equations with rational coefficients. II, Physica D 2 (1981) 407–448.
[10] R. Lin, R. Conte, M. Musette, On the Lax pairs of the continuous and discrete sixth Painlevé equations, J. Nonlinear Math. Phys. 10 (Supp. 2) (2003) 107–118.
[11] J. Malmquist, Sur les équations différentielles du second ordre dont l'intégrale générale a ses points critiques fixes, Arkiv Math., Astr., Fys. 17 (1922–23) 1–89.
[12] D.P. Novikov, The 2 × 2 matrix Schlesinger system and the Belavin–Polyakov–Zamolodchikov system, Teor. Mat. Fiz. 161 (2009) 191–203; Theor. Math. Phys. 161 (2009) 1485–1496.
[13] P. Painlevé, Sur les équations différentielles du second ordre à points critiques fixes, C. R. Acad. Sci. Paris 143 (1906) 1111–1117.
[14] H. Poincaré, Sur les groupes des équations linéaires, Acta Math. 4 (1883) 201–312; Reprinted: Oeuvres, tome II, Gauthier-Villars, Paris, 1951–1956, pp. 300–401.
[15] B.I. Suleimanov, Hamiltonian property of the Painlevé equations and the method of isomonodromic deformations, Differ. Uravn. 30 (1994) 791–796; English: Differential Equations 30 (1994) 726–732.
[16] C.A. Tracy, H. Widom, Level-spacing distributions and the Airy kernel, Commun. Math. Phys. 151 (1994) 151–174.
[17] A. Zabrodin, A. Zotov, Quantum Painlevé–Calogero correspondence, J. Math. Phys. 53 (2012) 073507;
A. Zabrodin, A. Zotov, Quantum Painlevé–Calogero correspondence for Painlevé VI, J. Math. Phys. 53 (2012) 073508, http://arxiv.org/abs/1107.5672.