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Some applications of the r-Whitney numbers

Quelques applications des nombres r-Whitney

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ABSTRACT

The main object of this paper is to give an application of the r-Whitney numbers to the values at rational arguments of the high-order Bernoulli and Euler polynomials. The obtained formulas generalize the known formulas of the Bernoulli numbers of both kinds.

RÉSUMÉ

Le but de ce papier est de présenter une application des nombres r-Whitney aux valeurs des polynômes de Bernoulli et d'Euler aux points rationnels. Les résultats obtenus généralisent les formules connues des nombres de Bernoulli des deux espèces.

1. Introduction

The r-Whitney numbers $w_{m,r}(n,k)$ and $W_{m,r}(n,k)$ of the first and the second kinds, respectively, can be defined as coefficients when we write $m^n(x)_n$ in the basis $\{(mx+r)^k;\ k=0,\dots n\}$ and $(mx+r)^n$ in the basis $\{(x)_k;\ k=0,\dots,n\}$, i.e.

$$m^n(x)_n=\sum_{k=0}^{n} w_{m,r}(n,k)(mx+r)^k \quad \text{and} \quad (mx+r)^n=\sum_{k=0}^{n} m^k W_{m,r}(n,k)(x)_k,$$

where $(\alpha)_n=\alpha(\alpha-1)\cdots(\alpha-n+1)$ if $n\geq 1$ and $(\alpha)_0=1$.

As it is known, these numbers have exponential generating functions as

$$\sum_{n\geq k} w_{m,r}(n,k)\frac{t^n}{n!}=\frac{1}{k!}\left(\frac{\ln(1+mt)}{m}\right)^k(1+mt)^{-\frac{r}{m}}, \tag{1}$$

$$\sum_{n\geq k} W_{m,r}(n,k)\frac{t^n}{n!}=\frac{1}{k!}\left(\frac{\exp(mt)-1}{m}\right)^k\exp(rt). \tag{2}$$

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The r-Whitney numbers $w_{m,r}(n,k)$ of the first kind give back the absolute Stirling numbers of the first kind $\left[{n \atop k}\right]$, the absolute r-Stirling numbers of the first kind $\left[{n \atop k}\right]_r$ and the Whitney numbers of the first kind $w_m(n,k)$, respectively,

$$w_{1,0}(n,k) = (-1)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}, \qquad w_{1,r}(n,k) = (-1)^{n-k}\begin{bmatrix} n+r \\ k+r \end{bmatrix}_r \quad \text{and} \quad w_{m,0}(n,k) = w_m(n,k).$$

The r-Whitney numbers $W_{m,r}(n,k)$ of the second kind give back the Stirling numbers of the second kind $\left\{{n \atop k}\right\}$, the r-Stirling numbers of the second kind $\left\{{n \atop k}\right\}_r$ and the Whitney numbers of the second kind $W_m(n,k)$, respectively,

$$W_{1,0}(n,k) = \begin{Bmatrix} n \\ k \end{Bmatrix}, \qquad W_{1,r}(n,k) = \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r \quad \text{and} \quad W_{m,0}(n,k) = W_m(n,k).$$

For more information about these numbers, see [1–5,9,13].

Recently, Merca [8] gave links of these numbers to the symmetric functions, and Mihoubi et al. [10] gave some applications of the r-Stirling numbers to Bernoulli polynomials.

Motivated by these works, we show in this paper some links of the Whitney numbers of both kinds to the values at rational numbers of the high-order Bernoulli and Euler polynomials. Indeed, we prove that the following identities (see [6, p. 560] and [11])

$$B_n = \sum_{j=0}^{n} (-1)^j \frac{j!}{j+1}\begin{Bmatrix} n \\ j \end{Bmatrix} \quad \text{and} \quad b_n = \frac{1}{n!}\sum_{j=0}^{n} \frac{(-1)^{n-j}}{j+1}\begin{bmatrix} n \\ j \end{bmatrix}, \quad n \geq 0. \tag{3}$$

can be generalized on the values at rational numbers of $B_n^{(k)}(x)$ and $b_n^{(k)}(x)$, where $B_n^{(\alpha)}(x)$ and $b_n^{(\alpha)}(x)$ are, respectively, the n-th high-order Bernoulli polynomials $B_n^{(\alpha)}(x)$ and $b_n^{(\alpha)}(x)$ of the first and the second kinds, $B_n := B_n^{(1)}(0)$ and $b_n := b_n^{(1)}(0)$; see for instance [7,12,14,15]. Recall that these polynomials are defined by their exponential generating functions as

$$\sum_{n \geq 0} B_n^{(\alpha)}(x)\frac{t^n}{n!} = \left(\frac{t}{\exp(t)-1}\right)^{\alpha}\exp(xt), \tag{4}$$

$$\sum_{n \geq 0} b_n^{(\alpha)}(x)t^n = \left(\frac{t}{\ln(1+t)}\right)^{\alpha}(1+t)^x. \tag{5}$$

with $B_n^{(1)}(x) = B_n(x)$ and $b_n^{(1)}(x) = b_n(x)$, respectively, the Bernoulli polynomials of the first and second kinds. For an application to the n-th high order Euler polynomials $E_n^{(\alpha)}(x)$, we recall the exponential generating function of these polynomials:

$$\sum_{n \geq 0} E_n^{(\alpha)}(x)\frac{t^n}{n!} = \left(\frac{2}{\exp(t)+1}\right)^{\alpha}\exp(xt)$$

with $E_n(x) := E_n^{(1)}(x)$ the n-th Euler polynomial.

Below, we prove the following theorem.

Theorem 1. *For r, s, k, n, m be non-negative integers with $m \neq 0$, we have:*

$$B_n^{(k)}\left(\frac{r-s}{m}\right) = \frac{1}{m^n}\sum_{j=0}^{n}\binom{j+k}{k}^{-1} w_{m,s}(j+k,k)W_{m,r}(n,j), \tag{6}$$

$$b_n^{(k)}\left(\frac{r-s}{m}\right) = \frac{1}{m^n n!}\sum_{j=0}^{n}\binom{j+k}{k}^{-1} W_{m,r}(j+k,k)w_{m,s}(n,j) \tag{7}$$

and

$$B_n^{(-k)}\left(\frac{r}{m}\right) = \frac{1}{m^n}\binom{n+k}{k}^{-1} W_{m,r}(n+k,k), \tag{8}$$

$$b_n^{(-k)}\left(-\frac{r}{m}\right) = \frac{1}{m^n}\frac{k!}{(n+k)!} w_{m,r}(n+k,k). \tag{9}$$

As consequences of Theorem 1, for $k = 1$, we obtain:

$$B_n\left(\frac{r-s}{m}\right)=\frac{1}{m^n}\sum_{j=0}^{n}\frac{1}{j+1}w_{m,s}(j+1,1)W_{m,r}(n,j),$$

$$b_n\left(\frac{r-s}{m}\right)=\frac{1}{m^{n+1}n!}\sum_{j=0}^{n}\frac{1}{j+1}\left((m+r)^{j+1}-r^{j+1}\right)w_{m,s}(n,j).$$

For $s=0$ and $k=1$ in Theorem 1, we get:

$$B_n\left(\frac{r}{m}\right)=\sum_{j=0}^{n}(-1)^j\frac{j!}{j+1}\frac{W_{m,r}(n,j)}{m^{n-j}},$$

$$b_n\left(\frac{r}{m}\right)=\frac{1}{m^{n+1}n!}\sum_{j=0}^{n}\frac{1}{j+1}\left((m+r)^{j+1}-r^{j+1}\right)w_m(n,j)$$

and for $r=0$ and $k=1$ in Theorem 1 we get:

$$B_n\left(-\frac{s}{m}\right)=\sum_{j=0}^{n}\frac{1}{j+1}\left\{ n \atop j \right\}\frac{w_{m,s}(j+1,1)}{m^j},$$

$$b_n\left(-\frac{s}{m}\right)=\frac{1}{n!}\sum_{j=0}^{n}\frac{1}{j+1}\frac{w_{m,s}(n,j)}{m^{n-j}}.$$

For $m=1$ in Theorem 1 we get the following corollary.

Corollary 2. *For r,s,k,n be non-negative integers, we have:*

$$B_n^{(k)}(r-s)=\sum_{j=0}^{n}(-1)^j\binom{j+k}{k}^{-1}\left[j+k+s \atop k+s \right]_s\left\{ n+r \atop j+r \right\}_r, \tag{10}$$

$$b_n^{(k)}(r-s)=\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{j+k}{k}^{-1}\left\{ j+k+r \atop k+r \right\}_r\left[n+s \atop j+s \right]_s. \tag{11}$$

As consequences of Corollary 2, for $k=1$ and from the known identities

$$\left[n+s \atop 1+s \right]_s=n!H_n^{(s)}\quad\text{and}\quad\left\{ j+r+1 \atop r+1 \right\}_r=(r+1)^{j+1}-r^{j+1}$$

we get

$$B_n(r-s)=\sum_{j=0}^{n}(-1)^j j!H_{j+1}^{(s)}\left\{ n+r \atop j+r \right\}_r,$$

$$b_n(r-s)=\frac{1}{n!}\sum_{j=0}^{n}\frac{(-1)^{n-j}}{j+1}\left((r+1)^{j+1}-r^{j+1}\right)\left[n+s \atop j+s \right]_s,$$

where $H_n^{(s)}$ is the n-th hyperharmonic number of order s defined by

$$H_n^{(1)}=\sum_{j=1}^{n}\frac{1}{j},\qquad H_n^{(s)}=\sum_{j=1}^{n}H_n^{(s-1)},\quad s>1.$$

For $s=0$ and $k=1$ in Corollary 2, we get:

$$B_n(r)=\sum_{j=0}^{n}(-1)^j\frac{j!}{j+1}\left\{ n+r \atop j+r \right\}_r,$$

$$b_n(r)=\frac{1}{n!}\sum^{n}\frac{(-1)^{n-j}}{j+1}\left((r+1)^{j+1}-r^{j+1}\right)\left[n \atop j \right].$$

The last expression of $b_n(r)$ shows that we have:

$$b_n(x) = \frac{1}{n!}\sum_{j=0}^{n}\frac{(-1)^{n-j}}{j+1}\left((x+1)^{j+1} - x^{j+1}\right)\begin{bmatrix} n \\ j \end{bmatrix}, \quad x \in \mathbb{R}.$$

For $r = 0$ and $k = 1$ in Corollary 2 we get:

$$B_n(-s) = \sum_{j=0}^{n}(-1)^j j! H_{j+1}^{(s)}\left\{\begin{matrix} n \\ j \end{matrix}\right\},$$

$$b_n(-s) = \frac{1}{n!}\sum_{j=0}^{n}\frac{(-1)^{n-j}}{j+1}\begin{bmatrix} n+s \\ j+s \end{bmatrix}_s.$$

Other application to Euler polynomials is given by the following theorem.

Theorem 3. *For r, k, n, m be non-negative integers with $m \neq 0$, we have:*

$$E_n^{(\alpha)}\left(\frac{r}{m}\right) = \frac{1}{m^n}\sum_{j=0}^{n}\left(-\frac{m}{2}\right)^j W_{m,r}(n, j)(\alpha)^j.$$

In particular, for $\alpha = 1$ or $m = 1$, we obtain:

$$E_n\left(\frac{r}{m}\right) = \frac{1}{m^n}\sum_{j=0}^{n}\left(-\frac{m}{2}\right)^j j! W_{m,r}(n, j), \qquad E_n^{(\alpha)}(r) = \sum_{j=0}^{n}\left(-\frac{1}{2}\right)^j \left\{\begin{matrix} n+r \\ j+r \end{matrix}\right\}_r (\alpha)^j,$$

where $(\alpha)^n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ if $n \geq 1$ and $(\alpha)^0 = 1$.

2. The proofs

Proof of Theorem 1. To prove (6), use the generating functions (1), (2) and (5) to obtain:

$$\begin{aligned}
\sum_{n\geq 0} B_n^{(k)}\left(\frac{r-s}{m}\right)\frac{t^n}{n!} &= \left(\frac{t}{\exp(t)-1}\right)^k \exp\left(\left(\frac{r-s}{m}\right)t\right) \\
&= k!\frac{\exp(r\frac{t}{m})}{(\frac{\exp(t)-1}{m})^k}\left[\frac{(\ln(1+m(\frac{\exp(t)-1}{m})))^k}{m^k k!(1+m(\frac{\exp(t)-1}{m}))^{\frac{s}{m}}}\right] \\
&= m^k k! \exp\left(r\frac{t}{m}\right)\sum_{j\geq k} w_{m,s}(j,k)\frac{1}{j!}\left(\frac{\exp(t)-1}{m}\right)^{j-k} \\
&= \sum_{j\geq 0}\binom{j+k}{k}^{-1} w_{m,s}(j+k,k)\frac{1}{j!}\left(\frac{\exp(m\frac{t}{m})-1}{m}\right)^j \exp\left(r\frac{t}{m}\right) \\
&= \sum_{j\geq 0}\binom{j+k}{k}^{-1} w_{m,s}(j+k,k)\sum_{n\geq j} W_{m,r}(n, j)\frac{(\frac{t}{m})^n}{n!} \\
&= \sum_{n\geq 0}\frac{t^n}{n!}\frac{1}{m^n}\sum_{j=0}^{n}\binom{j+k}{k}^{-1} w_{m,s}(j+k,k) W_{m,r}(n, j).
\end{aligned}$$

So, the desired identity follows by identification.

To prove (7), let us use the generating functions (1), (2) and (4) to obtain:

$$\begin{aligned}
\sum_{n\geq 0} b_n^{(k)}\left(\frac{r-s}{m}\right)t^n &= \left(\frac{t}{\ln(1+t)}\right)^k (1+t)^{\frac{r-s}{m}} \\
&= k!\frac{(1+t)^{-\frac{s}{m}}}{(\frac{\ln(1+t)}{m})^k}\left[\frac{1}{k!}\left(\frac{\exp(m\frac{\ln(1+t)}{m})-1}{m}\right)^k \exp\left(r\frac{\ln(1+t)}{m}\right)\right] \\
&= k!(1+t)^{-\frac{s}{m}}\sum_{j\geq k} W_{m,r}(j,k)\frac{1}{j!}\left(\frac{\ln(1+t)}{m}\right)^{j-k}
\end{aligned}$$

$$= \sum_{j \geq 0} \binom{j+k}{k}^{-1} W_{m,r}(j+k,k)\left(\frac{1}{j!}\left(\frac{\ln(1+m\frac{t}{m})}{m}\right)^j \left(1+m\frac{t}{m}\right)^{-\frac{s}{m}}\right)$$
$$= \sum_{j \geq 0} \binom{j+k}{k}^{-1} W_{m,r}(j+k,k) \sum_{n \geq j} w_{m,s}(n,j)\frac{(\frac{t}{m})^n}{n!}$$
$$= \sum_{n \geq 0} \frac{t^n}{n!}\frac{1}{m^n} \sum_{j=0}^{n} \binom{j+k}{k}^{-1} W_{m,r}(j+k,k) w_{m,s}(n,j).$$

To prove (8), let us use the generating functions (2) and (5) to obtain:

$$\sum_{n \geq 0} B_n^{(-k)}\left(\frac{r}{m}\right)\frac{(mt)^n}{n!} = t^{-k}\left(\frac{\exp(mt)-1}{m}\right)^k \exp(rt) = \sum_{n \geq 0} \binom{n+k}{k}^{-1} W_{m,r}(n+k,k)\frac{t^n}{n!}.$$

To prove (9), use the generating functions (1) and (4) to obtain:

$$\sum_{n \geq 0} b_n^{(k)}\left(-\frac{r}{m}\right)(mt)^n = t^{-k}\left(\frac{\ln(1+mt)}{m}\right)^k (1+mt)^{-\frac{r}{m}} = \sum_{n \geq 0} \binom{n+k}{k}^{-1} w_{m,r}(n+k,k)\frac{t^n}{n!}. \qquad \square$$

Proof of Theorem 3. The generating function of Euler polynomials gives:

$$\sum_{n \geq 0} E_n^{(\alpha)}\left(\frac{r}{m}\right)\frac{(mt)^n}{n!} = \left(\frac{2}{\exp(mt)+1}\right)^{\alpha} \exp(rt)$$
$$= \exp(rt)\left(\frac{\exp(mt)-1}{2}+1\right)^{-\alpha}$$
$$= \sum_{j \geq 0} \frac{(-\alpha)_j}{j!}\left(\frac{m}{2}\right)^j \left(\frac{\exp(mt)-1}{m}\right)^j \exp(rt)$$

and upon using the relation $(-\alpha)_j = (-1)^j (\alpha)^j$, we get:

$$\sum_{n \geq 0} E_n^{(\alpha)}\left(\frac{r}{m}\right)\frac{(mt)^n}{n!} = \sum_{j \geq 0}\left(-\frac{m}{2}\right)^j (\alpha)^j \sum_{n \geq j} W_{m,r}(n,j)\frac{t^n}{n!} = \sum_{n \geq 0}\frac{t^n}{n!}\sum_{j=0}^{n}\left(-\frac{m}{2}\right)^j W_{m,r}(n,j)(\alpha)^j$$

which gives, by identification, the desired identity.

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