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A note on the Zariski multiplicity conjecture

Note sur la conjecture de multiplicité de Zariski

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ABSTRACT

We prove that the algebraic multiplicities of two topologically equisingular isolated complex hypersurface singularities located at the origin are equal provided the continuous maps defining the topological right equivalence are Lipschitz on a generic real line segment departing from the origin.

RÉSUMÉ

On démontre que les multiplicités algébriques des singularités isolées de deux hypersurfaces complexes topologiquement équisingulières sont égales à condition que les applications qui définissent l'équivalence topologique à droite soient lipchitziennes sur un segment de droite réel, générique, contenant l'origine.

1. Introduction

Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be a germ of holomorphic function and V_f be the germ at the origin of the hypersurface defined by $f^{-1}(0)$. We suppose $0\in\mathbb{C}^n$ is an isolated singularity of the function. The *algebraic multiplicity* m_f of the germs of V_f or f is the order of vanishing of the function f at $0\in\mathbb{C}^n$ or equivalently is the order of the first nonzero leading term in the Taylor expansion of f

$$f=f_k+f_{k+1}+\cdots$$

where f_i is a homogeneous polynomial of degree i. Let $f,g:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be two germs of holomorphic functions at the origin. We say V_f and V_g are *topologically equisingular* if there is a germ of homeomorphism $\phi:(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ sending V_f onto V_g. More precisely, there are neighborhoods U and U' of $0\in\mathbb{C}^n$ such that f and g are defined and a homeomorphism $\phi:U\to U'$ such that $\phi(f^{-1}(0)\cap U)=g^{-1}(0)\cap U'$ and $\phi(0)=0$.

The Zariski multiplicity conjecture. *(See [10].) Does topological equisingularity of germs of complex hypersurface singularities imply equimultiplicity?*

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For a survey on this conjecture, we refer the reader to [3,9]. Before stating our result, let us recall some facts on complex hypersurface singularities and introduce some notation.

For every $z \in \mathbb{C}^n \setminus \{0\}$ define $a_z : [0,1] \to \mathbb{C}^n$ as $a_z(t) := tz$. Set

$$I_z := a_z\big([0,1]\big). \tag{1}$$

Lemma 1. *Let $f(z) = f_k(z) + f_{k+1}(z) + \cdots$. There are infinitely many points $w \in \mathbb{C}^n \setminus \{0\}$ such that*

$$I_w \cap V_f = I_w \cap V_{f_k} = \{0\}. \tag{$\star$}$$

Proof. By a linear change of coordinates we may assume that the homogeneous polynomial f_k has the term z_n^k. Then the intersection of the axis oz_n with the $f_k^{-1}(0)$ is the origin. Set $z = (0, \ldots, 0, z_n)$ in the expansion of f. We get the following polynomial (see below)

$$f(0, \ldots, 0, z_n) = z_n^k + P(z_n),$$

for some polynomial P, which has finitely many roots. Therefore we may choose infinitely many segments I_w in $\{(0, \ldots, 0, z_n) | z_n \in \mathbb{C}\}$ that satisfy ($\star$). □

By a classical result of Samuel [8] and independently by Mather (unpublished [5, 10.8]), we may assume that f and g are polynomials. Moreover, we may choose such polynomials by cutting the Taylor expansion of germs somewhere. Therefore the algebraic multiplicities do not change. By a remarkable result due to King [4] (for $n \neq 3$) and Perron [6] (for $n = 3$), g is right equivalent to f or $\bar{f}$, the complex conjugate of f. Therefore there is a germ of homeomorphism $\phi = (\phi_1(z), \ldots, \phi_n(z)) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\|f(z)\| = \|g \circ \phi(z)\|$.

Main theorem. *Let f and g be two germs of holomorphic functions at the origin with the Taylor expansions:*

$$f(z) = f_k(z) + f_{k+1}(z) + \cdots,$$
$$g(z) = g_l(z) + g_{l+1}(z) + \cdots.$$

Suppose that there is a germ of homeomorphism $\phi = (\phi_1(z), \ldots, \phi_n(z)) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\|f(z)\| = \|g \circ \phi(z)\|$. Assume that there is a real line segment I_w satisfying ($\star$) in Lemma 1 such that $\phi(z)/z$ is bounded on I_w. Then $l \leq k$.

An immediate consequence of the main theorem is the following corollary.

Corollary. *The algebraic multiplicities of two topologically equisingular isolated complex hypersurface singularities located at the origin are equal provided the continuous maps defining the topological right equivalence are Lipschitz on a generic real line segment departing from the origin.*

The corollary improves results of Risler–Trotman [7] and Comte–Milman–Trotman [2], where authors in [7] consider Bi-Lipschitz equivalent germs and in [2] consider right equivalent germs via ϕ such that $\phi(z)/z$ and $z/\phi(z)$ are bounded for all z near to the origin.

2. Proof of the main theorem

Proof of the main theorem. On the contrary, suppose $l > k$.

Set $z = tw$ in $\|f(z)\| = \|g \circ \phi(z)\|$ where $t \in [0,1]$. Therefore we obtain:

$$\big\|f_k(tw) + f_{k+1}(tw) + \cdots\big\| = \big\|g\big(\phi_1(tw), \ldots, \phi_n(tw)\big)\big\|,$$
$$\big\|t^k\big[f_k(w) + tf_{k+1}(w) + \cdots\big]\big\| = \big\|g\big(\phi_1(tw), \ldots, \phi_n(tw)\big)\big\|,$$
$$\big\|f_k(w) + tf_{k+1}(w) + \cdots\big\| = \left\|\frac{g(\phi_1(tw), \ldots, \phi_n(tw))}{t^k}\right\|.$$

Define $u_i : [0,1] \to \mathbb{C}, \forall i \in \{1, \cdots, n\}$ as the following:

$$u_i(t) := \phi_i(tw). \tag{2}$$

By the Weierstrass approximation theorem, for every $i \in \{1, \cdots, n\}$ there is a sequence of polynomials $\{P_m^i(t, w)\}_m$ which is uniformly convergent to $u_i(t)$. Note that w is fixed. We may choose P_m^i such that $P_m^i(0, w) = 0$ since $u_i(0) = 0$. By a constructive proof of the Weierstrass approximation theorem via Bernstein polynomials [1], we may consider polynomials as follows:

$$P_m^i(t,w)=\sum_{j=1}^{m}u_i(j/m)b_{j,m}(t), \tag{3}$$

where

$$b_{j,m}(t):=\binom{m}{j}t^j(1-t)^{m-j},\quad j=1,\dots,m \tag{4}$$

are Bernstein polynomials.

Define a sequence of polynomials as follows:

$$G_m(t,w):=g\big(P_m^1(t,w),\dots,P_m^n(t,w)\big). \tag{5}$$

The sequence $\{G_m\}_m$ is uniformly convergent to $g\circ\phi$ and

$$\begin{aligned}&\lim_{m\to\infty}\big\|G_m(t,w)\big\|=\big\|f(tw)\big\|,\\&\lim_{m\to\infty}\frac{\|G_m(t,w)\|}{t^k}=\big\|f_k(w)+tf_{k+1}(w)+\cdots\big\|\quad(t>0),\\&\lim_{t\to0}\lim_{m\to\infty}\frac{\|G_m(t,w)\|}{t^k}=\lim_{t\to0}\big\|f_k(w)+tf_{k+1}(w)+\cdots\big\|.\end{aligned} \tag{6}$$

Set

$$A:=\big\|f_k(w)\big\|. \tag{7}$$

$A>0$ since $w\notin f_k^{-1}(0)$.

Therefore

$$\big\|f_k(w)+tf_{k+1}(w)+\cdots\big\|>A/2 \tag{8}$$

for t small enough.

Suppose that

$$\begin{aligned}g(z)&=g_l(z)+g_{l+1}(z)+\cdots+g_{l+r}(z)\\&=\sum_{|\alpha|=l}C_{\alpha,l}z^\alpha+\cdots+\sum_{|\alpha|=l+r}C_{\alpha,l+r}z^\alpha,\end{aligned}$$

for some $r\in\mathbb{N}\cup\{0\}$ and some coefficients $C_{\alpha,l},\dots,C_{\alpha,l+r}\in\mathbb{C}$, where $|\alpha|:=\alpha_1+\cdots+\alpha_n$ and $z^\alpha:=z_1^{\alpha_1}\dots z_n^{\alpha_n}$.

Therefore

$$\begin{aligned}G_m(t,w)&=g\big(P_m^1(t,w),\dots,P_m^n(t,w)\big)\\&=\sum_{|\alpha|=l}C_{\alpha,l}\big(P_m^1(t,w)\big)^{\alpha_1}\dots\big(P_m^n(t,w)\big)^{\alpha_n}+\cdots\\&\quad+\sum_{|\alpha|=l+r}C_{\alpha,l+r}\big(P_m^1(t,w)\big)^{\alpha_1}\dots\big(P_m^n(t,w)\big)^{\alpha_n}.\end{aligned}$$

By substituting $P_m^i(t,w)=\sum_{j=1}^m u_i(j/m)b_{j,m}(t)$, $u_i(j/m)=\phi_i(\frac{jw}{m})$ and $b_{j,m}(t):=\binom{m}{j}t^j(1-t)^{m-j}$, $j=1,\dots,m$, we get the following:

$$\begin{aligned}G_m(t,w)&=\sum_{|\alpha|=l}C_{\alpha,l}\left(\sum_{j=1}^m\binom{m}{j}\phi_1\left(\frac{jw}{m}\right)t^j(1-t)^{m-j}\right)^{\alpha_1}\dots\\&\quad\times\left(\sum_{j=1}^m\binom{m}{j}\phi_n\left(\frac{jw}{m}\right)t^j(1-t)^{m-j}\right)^{\alpha_n}\\&\quad+\cdots\\&\quad+\sum_{|\alpha|=l+r}C_{\alpha,l+r}\left(\sum_{j=1}^m\binom{m}{j}\phi_1\left(\frac{jw}{m}\right)t^j(1-t)^{m-j}\right)^{\alpha_1}\dots\\&\quad\times\left(\sum_{j=1}^m\binom{m}{j}\phi_n\left(\frac{jw}{m}\right)t^j(1-t)^{m-j}\right)^{\alpha_n}.\end{aligned}$$

By the assumption, $\phi(z)/z$ is bounded on I_w. Therefore there is a constant $K > 0$ such that $\|\phi(z)\| \le K\|z\|$ on I_w and we get:

$$\left\|\phi_i\left(\frac{jw}{m}\right)\right\| \le K\frac{j}{m}\|w\|, \quad \forall i = 1,\dots,n,\ \forall j = 1,\dots,m. \tag{9}$$

Hence

$$\begin{aligned}\left\|G_m(t,w)\right\| &\le \|w\|^l K^l \sum_{|\alpha|=l} \|C_{\alpha,l}\| \left(\sum_{j=1}^m \binom{m}{j}\left(\frac{j}{m}\right)t^j(1-t)^{m-j}\right)^{\alpha_1} \dots \\ &\quad \times \left(\sum_{j=1}^m \binom{m}{j}\left(\frac{j}{m}\right)t^j(1-t)^{m-j}\right)^{\alpha_n} \\ &\quad + \cdots \\ &\quad + \|w\|^{l+r} K \sum_{|\alpha|=l+r} \|C_{\alpha,l+r}\| \left(\left(\sum_{j=1}^m \binom{m}{j}\left(\frac{j}{m}\right)t^j(1-t)^{m-j}\right)\right)^{\alpha_1} \dots \\ &\quad \times \left(\sum_{j=1}^m \binom{m}{j}\left(\frac{j}{m}\right)t^j(1-t)^{m-j}\right)^{\alpha_n} \\ &= \|w\|^l K\left(\sum_{|\alpha|=l} \|C_{\alpha,l}\|\right)\left(\sum_{j=1}^m \binom{m}{j}\left(\frac{j}{m}\right)t^j(1-t)^{m-j}\right)^{l} \\ &\quad + \cdots \\ &\quad + \|w\|^{l+r} K^{l+r}\left(\sum_{|\alpha|=l+r} \|C_{\alpha,l+r}\|\right)\left(\left(\sum_{j=1}^m \binom{m}{j}\left(\frac{j}{m}\right)t^j(1-t)^{m-j}\right)\right)^{l+r}.\end{aligned}$$

Using the identity

$$\sum_{j=1}^m \binom{m}{j}\left(\frac{j}{m}\right)t^j(1-t)^{m-j} \equiv t \tag{10}$$

we get the following:

$$\begin{aligned}\left\|G_m(t,w)\right\| &\le t^l\|w\|^l K^l \sum_{|\alpha|=l} \|C_{\alpha,l}\| + t^{l+1}\|w\|^{l+1} K^{l+1} \sum_{|\alpha|=l+1} \|C_{\alpha,l+1}\| + \cdots \\ &\quad + t^{l+r}\|w\|^{l+r} K^{l+r} \sum_{|\alpha|=l+r} \|C_{\alpha,l+r}\|.\end{aligned}$$

Define

$$\begin{aligned}F(t,w) := \|w\|^l K^l \sum_{|\alpha|=l} \|C_{\alpha,l}\| + t\|w\|^{l+1} K^{l+1} \sum_{|\alpha|=l+1} \|C_{\alpha,l+1}\| + \cdots \\ + t^r\|w\|^{l+r} K^{l+r} \sum_{|\alpha|=l+r} \|C_{\alpha,l+r}\|.\end{aligned}$$

Hence

$$\left\|G_m(t,w)\right\| \le t^l F(t,w). \tag{11}$$

By dividing both sides of (11) by t^k, we get:

$$\frac{\|G_m(t,w)\|}{t^k} \le t^{l-k} F(t,w). \tag{12}$$

Note that $l - k > 0$ by the contrary assumption. If t is chosen small enough, then the right-hand side of (12) is less than $A/2$. See the definition of A in (7). Therefore

$$\frac{\|G_m(t, w)\|}{t^k} < A/2. \tag{13}$$

The relations (6), (8) and (13) yield a contradiction. □

3. Toward the classification of the topological type of isolated singularities

Regarding the classification of the topological type of isolated singularities, we may assume that the functions called into question are polynomials. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of polynomial at the origin of $\mathbb{C}^n$ and

$$f(z) = f_k(z) + f_{k+1}(z) + \cdots + f_{k+r}(z),$$

be its Taylor expansion such that $0 \in \mathbb{C}^n$ is an isolated singularity for f and $r \in \mathbb{N} \cup \{0\}$ is the minimum number with this property. More precisely, either $r = 0$ or $0 \in \mathbb{C}^n$ is not an isolated singularity for $f(z) = f_k(z) + f_{k+1}(z) + \ldots + f_{k+r'}(z)$ for all $0 \le r' < r$.

Note that if $r = 0$ then $f(z) = f_k(z)$ and the hypersurface defined by f is topologically equivalent to the Fermat hypersurface defined by $z_1^k + \ldots + z_n^k$.

If we assume the Zariski multiplicity conjecture, then k is a topological invariant. With the above notation and preparation, we pose the following question.

Question. *Assuming the Zariski multiplicity conjecture, is the number r a topological invariant?*

Note that the Zariski multiplicity conjecture implies that $r = 0$ is a topological invariant. More precisely, if 0 is an isolated singularity of a homogeneous polynomial $f = f_k$ and the hypersurface defined by f is topologically equivalent to another hypersurface defined by g, then 0 is also an isolated singularity for the first nonzero homogeneous polynomial in the Taylor expansion of g and g is topologically equivalent to g_k. See [9, Example 4.7]. Therefore the above question has an affirmative answer in the case of $r = 0$. Notice that the Zariski multiplicity conjecture has a positive answer in the case of germs of isolated singular curves ($n = 2$) [11].

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