Number theory/Algebraic geometry

The Arithmetic Site ☆

Le Site arithmétique

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A B S T R A C T

We show that the non-commutative geometric approach to the Riemann zeta function has an algebraic geometric incarnation: the "Arithmetic Site". This site involves the tropical semiring $\bar{\mathbb{N}}$ viewed as a sheaf on the topos $\widehat{\mathbb{N}^{\times}}$ dual to the multiplicative semigroup of positive integers. We realize the Frobenius correspondences in the square of the "Arithmetic Site".

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R É S U M É

Le « Site arithmétique » est l'incarnation en géométrie algébrique de l'espace non commutatif, de nature adélique, qui permet d'obtenir la fonction zêta de Riemann comme fonction de dénombrement de Hasse–Weil. Ce site est construit à partir du semi-anneau tropical $\bar{\mathbb{N}}$ vu comme un faisceau sur le topos $\widehat{\mathbb{N}^{\times}}$ dual du semigroupe multiplicatif des entiers positifs. Nous réalisons les correspondances de Frobenius dans le carré du « Site arithmétique ».

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1. Introduction

We unveil the "Arithmetic Site" as a ringed topos deeply related to the non-commutative geometric approach to RH. The topos is the presheaf topos $\widehat{\mathbb{N}^{\times}}$ of functors from the multiplicative semigroup $\mathbb{N}^{\times}$ of positive integers to the category of sets. The structure sheaf is a sheaf of semirings of characteristic 1 and (as an object of the topos) is the tropical semiring $\bar{\mathbb{N}} := (\mathbb{N} \cup \infty, \inf, +)$, $\mathbb{N} = \mathbb{Z}_{\geq 0}$, on which the semigroup $\mathbb{N}^{\times}$ acts by multiplication. We prove that the set of points of the arithmetic site $(\widehat{\mathbb{N}^{\times}}, \bar{\mathbb{N}})$ over the maximal compact subring $[0, 1]_{\max} \subset \mathbb{R}_{+}^{\max}$ of the tropical semifield is the non-commutative space $\mathbb{Q}^{\times} \backslash \mathbb{A}_{\mathbb{Q}} / \hat{\mathbb{Z}}^{*}$, quotient of the adèle class space of $\mathbb{Q}$ by the action of the maximal compact subgroup $\hat{\mathbb{Z}}^{*}$ of the idèle class group. In [5,6] it was shown that the action of $\mathbb{R}_{+}^{*}$ on $\mathbb{Q}^{\times} \backslash \mathbb{A}_{\mathbb{Q}} / \hat{\mathbb{Z}}^{*}$ yields the counting distribution whose Hasse–Weil

zeta function is the complete Riemann zeta function. This result is now applied to the arithmetic site to show that its Hasse–Weil zeta function is the complete Riemann zeta function. The action of $\mathbb{R}_+^*$ on $\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}/\hat{\mathbb{Z}}^*$ indeed corresponds to the action of the Frobenius automorphisms $\mathrm{Fr}_\lambda \in \mathrm{Aut}(\mathbb{R}_+^{\max})$, $\lambda \in \mathbb{R}_+^*$, on the points of $(\widehat{\mathbb{N}^\times}, \bar{\mathbb{N}})$ over $[0,1]_{\max} \subset \mathbb{R}_+^{\max}$. The square of the arithmetic site over the semifield $\mathbb{B} = (\{0,1\}, \max, \times)$ has an unreduced and a reduced version. In both cases the underlying topos is $\widehat{\mathbb{N}^{\times 2}}$. The structure sheaf in the unreduced case is $\bar{\mathbb{N}} \otimes_\mathbb{B} \bar{\mathbb{N}}$ and in the reduced case it is the multiplicatively cancellative semiring canonically associated with $\bar{\mathbb{N}} \otimes_\mathbb{B} \bar{\mathbb{N}}$. We determine this latter semiring as the semiring $\mathrm{Conv}_\geq(\mathbb{N} \times \mathbb{N})$ of Newton polygons with the operations of convex hull of the union and sum. On both versions there is a canonical action of $\mathbb{N}^{\times 2}$ by endomorphisms $\mathrm{Fr}_{n,m}$. By composing this action with the diagonal (given by the product μ), one obtains the Frobenius correspondences $\Psi(\lambda) = \mu \circ \mathrm{Fr}_{n,m}$ for rational values $\lambda = n/m$. The Frobenius correspondences $\Psi(\lambda)$ for arbitrary positive real numbers λ are realized as curves in the square obtained from the rational case using diophantine approximation. Finally we determine the composition law of these correspondences and show that it is given by the product law in $\mathbb{R}_+^*$ with a subtle nuance in the case of two irrational numbers whose product is rational.

This note provides the algebraic geometric space underlying the non-commutative approach to RH. It gives a geometric framework reasonably suitable to transpose the conceptual understanding of the Weil proof in finite characteristic as in [7]. This translation would require in particular an adequate version of the Riemann–Roch theorem in characteristic 1.

2. The arithmetic site

Given a small category $\mathcal{C}$, we denote by $\hat{\mathcal{C}}$ the topos of contravariant functors from $\mathcal{C}$ to the category of sets. We let $\mathbb{N}^\times$ be the category with a single object $*$, $\mathrm{End}(*) = \mathbb{N}^\times$.

Definition 2.1. We define the *arithmetic site* $(\widehat{\mathbb{N}^\times}, \bar{\mathbb{N}})$ as the topos $\widehat{\mathbb{N}^\times}$ endowed with the *structure sheaf* $\bar{\mathbb{N}} := (\mathbb{N} \cup \infty, \inf, +)$ viewed as a semiring in the topos.

Notice that $\widehat{\mathbb{N}^\times} \simeq \mathrm{Sh}(\mathbb{N}^\times, J)$, where J is the chaotic topology on $\mathbb{N}^\times$ ([1] Exposé IV, 2.6).

2.1. The points of the topos $\widehat{\mathbb{N}^\times}$

A point of a topos $\mathcal{T}$ is defined as a geometric morphism from the topos of sets to $\mathcal{T}$ [1,8].

Theorem 2.2.

(i) *The category of points of the topos $\widehat{\mathbb{N}^\times}$ is canonically equivalent to the category of totally ordered groups isomorphic to non-trivial subgroups of $(\mathbb{Q}, \mathbb{Q}_+)$, and injective morphisms of ordered groups.*

(ii) *Let $\mathbb{A}_f$ be the ring of finite adèles of $\mathbb{Q}$. The space of isomorphism classes of points of $\widehat{\mathbb{N}^\times}$ is canonically isomorphic to the double quotient $\mathbb{Q}_+^\times\backslash\mathbb{A}_f/\hat{\mathbb{Z}}^*$ where $\mathbb{Q}_+^\times$ acts by multiplication on $\mathbb{A}_f$.*

We denote by $\mathbb{F} = \mathbb{Z}_{\max}$ the semifield of fractions of the semiring $\bar{\mathbb{N}}$.

Corollary 2.3. *The category of points of the topos $\widehat{\mathbb{N}^\times}$ is equivalent to the category of algebraic extensions of the semifield $\mathbb{F} = \mathbb{Z}_{\max}$ i.e. of extensions: $\mathbb{F} \subset K \subset \mathbb{Q}_{\max}$. The morphisms are the injective morphisms of semifields.*

2.2. The structure sheaf $\bar{\mathbb{N}}$

The next result provides an explicit description of the semiring structure inherited automatically by the stalks of the sheaf $\bar{\mathbb{N}}$ on the topos $\widehat{\mathbb{N}^\times}$.

Theorem 2.4. *At the point of the topos $\widehat{\mathbb{N}^\times}$ associated with the intermediate semifield $\mathbb{F} \subset K \subset \mathbb{Q}_{\max}$, the stalk of the structure sheaf $\mathcal{O} := \bar{\mathbb{N}}$ is the semiring $\mathcal{O}_K := \{r \in K \mid r \vee 1 = 1\}$ where $\vee$ denotes addition.*

2.3. The points of the arithmetic site $(\widehat{\mathbb{N}^\times}, \bar{\mathbb{N}})$ over $[0,1]_{\max}$

The following definition provides the notion of point of the arithmetic site over a local semiring.

Definition 2.5. Let R be a local semiring. Then a point of $(\widehat{\mathbb{N}^\times}, \bar{\mathbb{N}})$ over R is a pair of a point p of $\widehat{\mathbb{N}^\times}$ and a local morphism of semirings $f_p^\# : \mathcal{O}_p \to R$.

We let $[0,1]_{\max} \subset \mathbb{R}_+^{\max}$ be the maximal compact sub-semiring of the tropical semifield $\mathbb{R}_+^{\max}$. The next crucial statement determines the interpretation of the space underlying the non-commutative geometric approach to RH in terms of algebraic geometry.

Theorem 2.6. *The points of the arithmetic site* $(\widehat{\mathbb{N}^\times}, \bar{\mathbb{N}})$ *over the local semiring* $[0, 1]_{\max}$ *form the quotient* $\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q} / \hat{\mathbb{Z}}^*$ *of the adèle class space of* $\mathbb{Q}$ *by the action of* $\hat{\mathbb{Z}}^*$. *The action of the Frobenius automorphisms* $\mathrm{Fr}_\lambda \in \mathrm{Aut}([0, 1]_{\max})$ *on these points corresponds to the action of the idèle class group (mod* $\hat{\mathbb{Z}}^*$) *on the above quotient of the adèle class space.*

Notice that the quotient $\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q} / \hat{\mathbb{Z}}^*$ is the disjoint union of the following two spaces:

(*i*) $\mathbb{Q}_+^\times \backslash \mathbb{A}_f / \hat{\mathbb{Z}}^*$ is the space of adèle classes whose Archimedean component vanishes. The corresponding points of the arithmetic site $(\widehat{\mathbb{N}^\times}, \bar{\mathbb{N}})$ are those which are defined over $\mathbb{B}$; they are given by the points of $\widehat{\mathbb{N}^\times}$ (Theorem 2.2).

(*ii*) $\mathbb{Q}_+^\times \backslash ((\mathbb{A}_f / \hat{\mathbb{Z}}^*) \times \mathbb{R}_+^*)$ is the space of adèle classes whose Archimedean component does not vanish. It is in canonical bijection with rank-one subgroups of $\mathbb{R}$ through the map

$$(a, \lambda) \mapsto \lambda H_a, \quad \forall a \in \mathbb{A}_f / \hat{\mathbb{Z}}^*, \ \lambda \in \mathbb{R}_+^*, \ H_a := \{q \in \mathbb{Q} \mid qa \in \hat{\mathbb{Z}}\}.$$

2.4. Hasse–Weil formula for the Riemann zeta function

In order to count the number of fixed points of the Frobenius action on points of $(\widehat{\mathbb{N}^\times}, \bar{\mathbb{N}})$ over $[0, 1]_{\max}$, we let $\vartheta_u \xi(x) = \xi(u^{-1}x)$ be the scaling action of the idèle class group $G = \mathrm{GL}_1(\mathbb{A}_\mathbb{Q}) / \mathrm{GL}_1(\mathbb{Q})$ on functions on the adèle class space $\mathbb{A}_\mathbb{Q} / \mathbb{Q}^*$ and use the trace formula [3,4,9] in the form ($\Sigma_\mathbb{Q}$ = places of $\mathbb{Q}$, d^*u multiplicative Haar measure)

$$\mathrm{Tr}_{\mathrm{distr}}\left(\int_G h(u)\vartheta_u \mathrm{d}^*u \right) = \sum_{v \in \Sigma_\mathbb{Q}} \int_{\mathbb{Q}_v^*} \frac{h(u^{-1})}{|1 - u|} \, \mathrm{d}^*u. \tag{1}$$

We apply (1) to test functions of the form $h(u) = g(|u|)$ where the support of g is contained in $(1, \infty)$ and $|u|$ is the module. The invariance of h under the kernel $\hat{\mathbb{Z}}^*$ of the module $G \to \mathbb{R}_+^*$ corresponds at the geometric level to taking the quotient of the adèle class space by the action of $\hat{\mathbb{Z}}^*$. Using Theorem 2.6 and [6], §2, one obtains the counting distribution $N(u)$, $u \in [1, \infty)$ associated with the Frobenius action on points of $(\widehat{\mathbb{N}^\times}, \bar{\mathbb{N}})$ over $[0, 1]_{\max}$.

Theorem 2.7. *The zeta function* ζ_N *associated by the equation*

$$\frac{\partial_s \zeta_N(s)}{\zeta_N(s)} = -\int_1^\infty N(u) \, u^{-s} \mathrm{d}^*u, \quad s \in \mathbb{C}, \ \Re(s) > 1 \tag{2}$$

with the counting distribution $N(u)$ *is the complete Riemann zeta function* $\zeta_\mathbb{Q}(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$.

In [5], Eq. (2) was shown (following a suggestion made in [11]) to be the limit, when $q \to 1$, of the Hasse–Weil formula for counting functions over finite fields $\mathbb{F}_q$.

3. The square of the arithmetic site

3.1. The unreduced square $(\widehat{\mathbb{N}^{\times 2}}, \bar{\mathbb{N}} \otimes_\mathbb{B} \bar{\mathbb{N}})$

Given a partially ordered set J, we let $\mathrm{Sub}_\geq(J)$ be the set of subsets $E \subset J$, which are hereditary, i.e. such that $x \in E \implies y \in E, \forall y \geq x$. Then $\mathrm{Sub}_\geq(J)$, endowed with the operation $E \oplus E' := E \cup E'$, is a $\mathbb{B}$-module. We refer to [10] for the general treatment of tensor products of semi-modules.

Proposition 3.1.

(*i*) *Let* $\mathbb{N} \times \mathbb{N}$ *be endowed with the partial order* $(a, b) \leq (c, d) \iff a \leq c \ \& \ b \leq d$. *Then one has a canonical isomorphism of* $\mathbb{B}$-*modules* $\bar{\mathbb{N}} \otimes_\mathbb{B} \bar{\mathbb{N}} \simeq \mathrm{Sub}_\geq(\mathbb{N} \times \mathbb{N})$.

(*ii*) *There exists on the* $\mathbb{B}$-*module* $S = \bar{\mathbb{N}} \otimes_\mathbb{B} \bar{\mathbb{N}}$ *a unique bilinear multiplication such that, using multiplicative notation where* q *is a formal variable, one has*

$$(q^a \otimes_\mathbb{B} q^b)(q^c \otimes_\mathbb{B} q^d) = q^{a+c} \otimes_\mathbb{B} q^{b+d}. \tag{3}$$

(*iii*) *The multiplication* (3) *turns* $\bar{\mathbb{N}} \otimes_\mathbb{B} \bar{\mathbb{N}}$ *into a semiring of characteristic* 1.

(*iv*) *The following formula defines an action of* $\mathbb{N}^\times \times \mathbb{N}^\times$ *by endomorphisms on* $\bar{\mathbb{N}} \otimes_\mathbb{B} \bar{\mathbb{N}}$

$$\mathrm{Fr}_{n,m}\left(\sum q^a \otimes_\mathbb{B} q^b \right) := \sum q^{na} \otimes_\mathbb{B} q^{mb}.$$

Definition 3.2. The *unreduced square* $(\widehat{\mathbb{N}^{\times 2}}, \bar{\mathbb{N}} \otimes_\mathbb{B} \bar{\mathbb{N}})$ of the arithmetic site $(\widehat{\mathbb{N}^\times}, \bar{\mathbb{N}})$ is the topos $\widehat{\mathbb{N}^{\times 2}}$ with the *structure sheaf* $\bar{\mathbb{N}} \otimes_\mathbb{B} \bar{\mathbb{N}}$, viewed as a semiring in the topos.
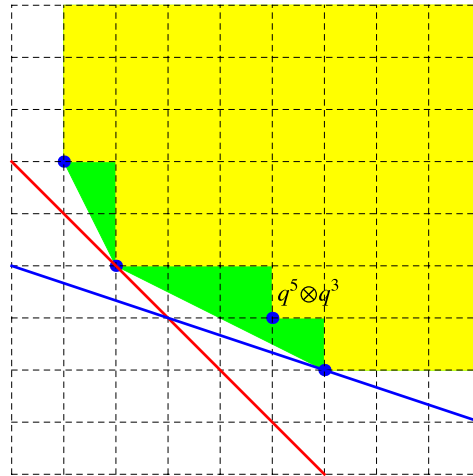
Fig. 1. Typical element $E \in \text{Sub}_\geq(\mathbb{N} \times \mathbb{N})$ (in yellow); its image under μ (red), under the Frobenius correspondence for $\lambda = \frac{1}{3}$ (blue) and under γ (green or yellow). Removing the element $q^5 \otimes_{\mathbb{B}} q^3$ does not alter the convex hull $\gamma(E)$. (For interpretation of references to color in this figure, the reader is referred to the web version of this article.)

3.2. The Frobenius correspondences

The product in the semiring $\overline{\mathbb{N}}$ yields a morphism of semirings $\mu : (\overline{\mathbb{N}} \otimes_{\mathbb{B}} \overline{\mathbb{N}}) \to \overline{\mathbb{N}}$, given on simple tensors by $\mu(q^a \otimes_{\mathbb{B}} q^b) = q^{a+b}$ (Fig. 1).

Proposition 3.3.

(i) *The range of the morphism $\mu \circ \text{Fr}_{n,m} : \overline{\mathbb{N}} \otimes_{\mathbb{B}} \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ only depends, up to the canonical isomorphism, on the ratio $r = n/m$. Assuming that n, m are relatively prime, this range contains the ideal*

$$\left\{ q^a \mid a \geq (n-1)(m-1) \right\} \subset \overline{\mathbb{N}}.$$

(ii) *Let $r = n/m$, $q \in (0, 1)$ and let $m_r : \overline{\mathbb{N}} \otimes_{\mathbb{B}} \overline{\mathbb{N}} \to \mathbb{R}_+^{\max}$ be given by*

$$m_r\left(\sum \left(q^{n_i} \otimes_{\mathbb{B}} q^{m_i} \right) \right) = q^\alpha, \quad \alpha = \inf(rn_i + m_i).$$

Up to the canonical isomorphism of their ranges, the morphisms $\mu \circ \text{Fr}_{n,m}$ and m_r are equal.

Proposition 3.3(*ii*) allows one to extend the definition of the Frobenius correspondence to arbitrary positive real numbers.

Proposition 3.4.

(i) *Let $\lambda \in \mathbb{R}_+^*$ and $q \in (0, 1)$ then the following formula defines a homomorphism*

$$\mathcal{F}(\lambda, q) : \overline{\mathbb{N}} \otimes_{\mathbb{B}} \overline{\mathbb{N}} \to \mathbb{R}_+^{\max}, \qquad \mathcal{F}(\lambda, q)\left(\sum \left(q^{n_i} \otimes_{\mathbb{B}} q^{m_i} \right) \right) = q^\alpha, \quad \alpha = \inf(\lambda n_i + m_i). \tag{4}$$

(ii) *The semiring $\mathcal{R}(\lambda) := \text{Im}(\mathcal{F}(\lambda, q))$ is independent, up to the canonical isomorphism, of $q \in (0, 1)$.*
(iii) *The semirings $\mathcal{R}(\lambda)$ and $\mathcal{R}(\lambda')$ are isomorphic if and only if $\lambda' = \lambda$ or $\lambda' = 1/\lambda$.*

3.3. The reduced square $(\widehat{\mathbb{N}^{\times 2}}, \text{Conv}_\geq(\mathbb{N} \times \mathbb{N}))$

Let R be a semiring without zero divisors and $\iota : R \to \text{Frac}R$ the canonical morphism to the semifield of fractions. It is not true in general that ι is injective (cf. [2]). We shall refer to $\iota(R)$ as the *reduced* semiring of R.

Definition 3.5. We let $\text{Conv}_\geq(\mathbb{N} \times \mathbb{N})$ be the set of closed convex subsets C of the quadrant $Q := \mathbb{R}_+ \times \mathbb{R}_+$ such that (i) $C + Q = C$ and (ii) the extreme points ∂C belong to $\mathbb{N} \times \mathbb{N} \subset Q$.

The set $\text{Conv}_\geq(\mathbb{N} \times \mathbb{N})$ is a semiring for the operations of convex hull of the union and sum.

Proposition 3.6.

(i) *The semiring $\text{Conv}_\geq(\mathbb{N} \times \mathbb{N})$ is multiplicatively cancellative.*

(ii) *The homomorphism* $\gamma : \bar{\mathbb{N}} \otimes_{\mathbb{B}} \bar{\mathbb{N}} \simeq \mathrm{Sub}_{\geq}(\mathbb{N} \times \mathbb{N}) \to \mathrm{Conv}_{\geq}(\mathbb{N} \times \mathbb{N})$ *given by convex hull is the same as the homomorphism* $\iota : \bar{\mathbb{N}} \otimes_{\mathbb{B}} \bar{\mathbb{N}} \to \iota(\bar{\mathbb{N}} \otimes_{\mathbb{B}} \bar{\mathbb{N}})$.

(iii) *Let R be a multiplicatively cancellative semiring and $\rho : \bar{\mathbb{N}} \otimes_{\mathbb{B}} \bar{\mathbb{N}} \to R$ a homomorphism of semirings such that $\rho^{-1}(\{0\}) = \{0\}$. Then there exists a unique semiring homomorphism $\rho' : \mathrm{Conv}_{\geq}(\mathbb{N} \times \mathbb{N}) \to R$ such that $\rho = \rho' \circ \gamma$.*

Definition 3.7. The *reduced square* $(\widehat{\mathbb{N}^{\times 2}}, \mathrm{Conv}_{\geq}(\mathbb{N} \times \mathbb{N}))$ of the arithmetic site $(\widehat{\mathbb{N}^{\times}}, \bar{\mathbb{N}})$ is the topos $\widehat{\mathbb{N}^{\times 2}}$ with the *structure sheaf* $\mathrm{Conv}_{\geq}(\mathbb{N} \times \mathbb{N})$, viewed as a semiring in the topos.

4. Composition of Frobenius correspondences

4.1. Reduced correspondences

Definition 4.1. A *reduced correspondence* over the arithmetic site $(\widehat{\mathbb{N}^{\times}}, \bar{\mathbb{N}})$ is given by a triple (R, ℓ, r) where R is a multiplicatively cancellative semiring, $\ell, r : \bar{\mathbb{N}} \to R$ are semiring morphisms such that $\ell^{-1}(\{0\}) = \{0\}$, $r^{-1}(\{0\}) = \{0\}$ and that R is generated by $\ell(\bar{\mathbb{N}}) r(\bar{\mathbb{N}})$.

By construction, cf. Proposition 3.4, the Frobenius correspondence gives a reduced correspondence:

$$\Psi(\lambda) := \big(R, \ell(\lambda), r(\lambda)\big), \quad R := \mathcal{R}(\lambda), \; \ell(\lambda)\big(q^n\big) := \mathcal{F}(\lambda, q)\big(q^n \otimes 1\big), \; r(\lambda)\big(q^n\big) := \mathcal{F}(\lambda, q)\big(1 \otimes q^n\big). \tag{5}$$

By (4) one gets that the elements of $\mathcal{R}(\lambda)$ are powers q^{α} where $\alpha \in \mathbb{N} + \lambda \mathbb{N}$ and that the morphisms $\ell(\lambda)$ and $r(\lambda)$ are described as follows:

$$\ell(\lambda)\big(q^n\big)q^{\alpha} = q^{\alpha + n\lambda}, \qquad r(\lambda)\big(q^n\big)q^{\alpha} = q^{\alpha + n}.$$

4.2. The composition of the correspondences $\Psi(\lambda) \circ \Psi(\lambda')$

The composition $\Psi(\lambda) \circ \Psi(\lambda')$ of the Frobenius correspondences is obtained as the left and right actions of $\bar{\mathbb{N}}$ on the reduced semiring of the tensor product $\mathcal{R}(\lambda) \otimes_{\bar{\mathbb{N}}} \mathcal{R}(\lambda')$. In order to state the general result, we introduce a variant Id_{ϵ} of the identity correspondence. We let $\mathrm{Germ}_{\epsilon=0}(\mathbb{R}_+^{\max})$ be the semiring of germs of continuous functions from a neighborhood of $0 \in \mathbb{R}$ to $\mathbb{R}_+^{\max}$, endowed with the pointwise operations. Let $\bar{\mathbb{N}}_{\epsilon}$ be the sub-semiring of $\mathrm{Germ}_{\epsilon=0}(\mathbb{R}_+^{\max})$ generated, for fixed $q \in (0,1)$, by q, and $\mathrm{Fr}_{1+\epsilon}(q) = q^{1+\epsilon}$. $\bar{\mathbb{N}}_{\epsilon}$ is independent, up to canonical isomorphism, of the choice of $q \in (0,1)$.

Definition 4.2. The tangential deformation of the identity correspondence is given by the triple $(\bar{\mathbb{N}}_{\epsilon}, \ell_{\epsilon}, r_{\epsilon})$ where $\ell_{\epsilon}(q^n) := \mathrm{Fr}_{1+\epsilon}(q^n)$ and $r_{\epsilon}(q^n) := q^n$, $\forall n \in \mathbb{N}$.

Theorem 4.3. *Let $\lambda, \lambda' \in \mathbb{R}_+^*$ such that $\lambda \lambda' \notin \mathbb{Q}$. The composition of the Frobenius correspondences is then given by*

$$\Psi(\lambda) \circ \Psi\big(\lambda'\big) = \Psi\big(\lambda \lambda'\big).$$

The same equality holds if λ and λ' are rational. When λ, λ' are irrational and $\lambda \lambda' \in \mathbb{Q}$,

$$\Psi(\lambda) \circ \Psi\big(\lambda'\big) = \Psi\big(\lambda \lambda'\big) \circ \mathrm{Id}_{\epsilon} = \mathrm{Id}_{\epsilon} \circ \Psi\big(\lambda \lambda'\big),$$

where Id_{ϵ} is the tangential deformation of the identity correspondence.

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