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# Perimeter of sets and *BMO*-type norms

## *Périmètre d'ensembles et normes de type BMO*

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ABSTRACT

The aim of this note is to announce some recent results showing that an isotropic variant of the *BMO*-type norm introduced in [3] can be related via a precise formula to the perimeter of sets.



RÉSUMÉ

Dans cette note, nous annonçons des résultats récents montrant qu'une variante isotrope de la norme de type *BMO* introduite dans [3] peut être reliée par une formule exacte au périmètre d'ensembles.



## 1. Introduction

In a very recent paper [3], the second and third author, in collaboration with P. Mironescu, introduced a new function space $B \subset L^1$ inspired by the celebrated *BMO* space of John–Nirenberg [5]. This space allowed them to show that a large class of $\mathbb{Z}$-valued functions must be constant. The analysis in [3] suggests the existence of a connection between the *BMO*-type seminorm and isoperimetric inequalities. We prove that indeed the relationship can be made very precise.

Since the concept of perimeter is isotropic, it is natural to introduce an isotropic version of the norm considered in [3] by defining, for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\mathrm{I}_\epsilon(f) := \epsilon^{n-1} \sup_{\mathcal{F}_\epsilon} \sum_{Q' \in \mathcal{F}_\epsilon} \fint_{Q'} \left| f(x) - \fint_{Q'} f \right| \mathrm{d}x, \tag{1.1}$$

where $\mathcal{F}_\epsilon$ denotes a collection of disjoint $\epsilon$-cubes $Q' \subset \mathbb{R}^n$ with *arbitrary* orientation and cardinality not exceeding $\epsilon^{1-n}$; the supremum is taken over all such collections.

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## 2. The principal theorem

Our main result is the following:

**Theorem 2.1.** *For any measurable set $A \subset \mathbb{R}^n$, $n \geq 1$, one has*

$$\lim_{\epsilon \to 0} \mathsf{I}_\epsilon(\mathbf{1}_A) = \frac{1}{2} \min\{1, \mathsf{P}(A)\}. \tag{2.1}$$

*In particular,* $\lim_\epsilon \mathsf{I}_\epsilon(\mathbf{1}_A) < 1/2$ *implies that $A$ has finite perimeter and* $\mathsf{P}(A) = 2 \lim_\epsilon \mathsf{I}_\epsilon(\mathbf{1}_A)$.

A few words about the strategy of proof when $n \geq 2$. Using the canonical decomposition in cubes, it is not too difficult to show the existence of dimensional constants $\xi_n, \eta_n > 0$ satisfying

$$\limsup_{\epsilon \downarrow 0} \mathsf{I}_\epsilon(\mathbf{1}_A) < \xi_n \quad \Longrightarrow \quad \mathsf{P}(A) \leq \eta_n \limsup_{\epsilon \downarrow 0} \mathsf{I}_\epsilon(\mathbf{1}_A) \tag{2.2}$$

for any measurable set $A \Subset (0,1)^n$. This idea can be very much refined, leading to the proof of the inequality $\liminf_\epsilon \mathsf{I}_\epsilon(\mathbf{1}_A) \geq 1/2$ whenever $P(A) = +\infty$. It is easy to see that $\mathsf{I}_\epsilon(\mathbf{1}_A) \leq 1/2$, and this proves our main result for sets of infinite perimeter. For sets of finite perimeter, the inequality $\leq$ in (2.1) relies on a kind of relative isoperimetric inequality in the cube with sharp constant, see [4] and [2], while the inequality $\geq$ relies on fine properties of sets of finite perimeter and a blow-up argument.

Using an obvious scaling argument, one may modify the bound on the cardinality and deduce that

$$\limsup_{\epsilon \downarrow 0} \sup_{\mathcal{F}_{\epsilon,M}} \epsilon^{n-1} \sum_{Q' \in \mathcal{F}_{\epsilon,M}} 2 \fint_{Q'} \left| \mathbf{1}_A(x) - \fint_{Q'} \mathbf{1}_A \right| \mathrm{d}x = \min\{M, \mathsf{P}(A)\} \quad \forall M > 0,$$

where $\mathcal{F}_{\epsilon,M}$ denotes a collection of $\epsilon$-cubes with arbitrary orientation and cardinality not exceeding $M\epsilon^{1-n}$.

A byproduct of our result is the following characterization of the perimeter:

$$\limsup_{\epsilon \downarrow 0} \sup_{\mathcal{H}_\epsilon} \epsilon^{n-1} \sum_{Q' \in \mathcal{H}_\epsilon} 2 \fint_{Q'} \left| \mathbf{1}_A(x) - \fint_{Q'} \mathbf{1}_A \right| \mathrm{d}x = \mathsf{P}(A), \tag{2.3}$$

where now $\mathcal{H}_\epsilon$ denotes a collection of disjoint $\epsilon$-cubes $Q' \subset \mathbb{R}^n$ with arbitrary orientation but no constraint on cardinality.

The above theorem can also be localized in Lipschitz domains, see [1, Section 4.1]: if $A$ is a measurable subset of a Lipschitz domain $\Omega$, then

$$\lim_{\epsilon \to 0} \mathsf{I}_\epsilon(\mathbf{1}_A, \Omega) = \frac{1}{2} \min\{1, \mathsf{P}(A, \Omega)\}, \tag{2.4}$$

where $\mathsf{I}_\epsilon(\mathbf{1}_A, \Omega)$ is a localized version of (1.1) in which one restricts the supremum over cubes contained in $\Omega$, and $\mathsf{P}(A, \Omega)$ denotes the perimeter of $A$ inside $\Omega$.

The detailed proofs are presented in [1].


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