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Partial differential equations

On the location of two blowup points on an annulus for the mean field equation

Sur l'emplacement de deux points d'explosion sur un annulus pour l'équation de champ moyen

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ABSTRACT

We consider the mean field equation on two-dimensional annular domains, and prove that if P_1 and P_2 are two blowup points of a blowing-up solution sequence of the equation, then we must have $P_1 = -P_2$.

RÉSUMÉ

Nous considérons l'équation de champ moyen sur les domaines annulaires à deux dimensions, et prouvons que, si P_1 et P_2 sont deux points d'explosion, alors nous devons avoir $P_1 = -P_2$.

1. Introduction

In this paper we consider the problem

$$-\Delta u = \lambda \frac{e^u}{\int_\Omega e^u dx} \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega, \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$ and $\lambda > 0$ is a parameter. Eq. (1) is known as the mean field equation and is considered to have relations with various fields of mathematical physics, such as Onsager's vortex theories, Chern–Simons–Higgs gauge theory, and so on. The interested readers should refer to the books by Tarantello [15], Yang [16], and the references therein. The possible blowing-up or non-compactness for a solution sequence of the problem have attracted many authors for more than two decades, and many efforts have been devoted to study such a critical phenomenon.

Now, thanks to the works by [14,3] and [13], we have the following description of the blowing-up solution sequences. Let u_n be a sequence of solutions to (1) for $\lambda = \lambda_n$ such that $\|u_n\|_{L^\infty(\Omega)}$ is not bounded from above, while $\lambda_n = O(1)$

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as $n \to \infty$. Then there exists a subsequence λ_n and a set $\mathcal{S} = \{a_1, \cdots, a_l\}$ with $a_i \in \Omega$, such that $\lambda_n \to 8\pi l$, $l \in \mathbb{N}$, and $\lambda_n \frac{e^{u_n}}{\int_\Omega e^{u_n} dx} \rightharpoonup 8\pi \sum_{i=1}^l \delta_{a_i}$ in the sense of measures. Moreover, each $a_i \in \mathcal{S}$ must satisfy the condition

$$\frac{1}{2}\nabla R(a_i) - \sum_{j=1, j\neq i}^{l} \nabla_x G(a_i, a_j) = \mathbf{0} \quad (i = 1, 2, \cdots, l), \tag{2}$$

where $G = G(x, y)$ is the Green function with pole $y \in \Omega$ subject to the Dirichlet boundary condition: $-\Delta_x G(x, y) = 2\pi\delta_y$ in Ω, $G(x, y)|_{x\in\partial\Omega} = 0$, and R is the Robin function defined as $R(y) = \lim_{x\to y}(\log|x - y|^{-1} - G(x, y))$. Therefore, the relation (2) can be considered as a characterization of the location of blowup points for (1).

On the other hand, several existence results of l-points blowing-up solutions to (1) have been found by several authors, see [8,7]. Their results can be summarized as follows.

Let $l \geq 1$ be an integer and set $\Delta = \{(x_1, \cdots, x_l) \in \Omega^l \mid x_i = x_j \text{ for some } i, j \in \{1, \cdots, l\}\}$, where $\Omega^l \subset \mathbb{R}^{2l}$ denotes an l-time products of Ω. Define $\mathcal{F} : \Omega^l \setminus \Delta \to \mathbb{R}$ as

$$\mathcal{F}(\xi_1, \cdots, \xi_l) = \sum_{i=1}^{l} R(\xi_i) - \sum_{\substack{i\neq j\\ 1\leq i,j\leq l}} G(\xi_i, \xi_j),$$

here, we agree that $\mathcal{F}(\xi) = R(\xi)$ for $\xi \in \Omega$ when $l = 1$. Note that the condition $\nabla_{(\xi_1,\cdots,\xi_l)}\mathcal{F}(a_1, \cdots, a_l) = 0$ is equivalent to (2) for $(a_1, \cdots, a_l) \in \Omega^l$. By these notations, let $(a_1, \cdots, a_l) \in \Omega^l \setminus \Delta$ be a "stable" critical point [8], or a "nontrivial" critical point [7] of $\mathcal{F}$, that is, $(a_1, \cdots, a_l)$ satisfies (2) and some additional "stability" or "nontriviality" condition is satisfied. Then there exists a sequence of solutions blowing up exactly at $\mathcal{S} = \{a_1, \cdots, a_l\}$. In particular, if the domain is not simply connected, there always exists a sequence of blowing-up solutions which blows up at l points on the domain for any $l \in \mathbb{N}$. Contrary to the above, we do not have any blowing-up solution sequence with multiple ($l \geq 2$) blow up points, if the domain is convex. This nonexistence of multiple blow up points holds true for several nonlinear problems other than (1), see [9]. The relationship between the location of blowup points and the geometry of the domain seems to be an interesting subject.

In this note, we turn to the study of the location of blowup points for the mean field equation (1). We concentrate to the case when Ω is an annulus. In this case, C.C. Chen and C.S. Lin [5] showed the following:

Theorem 1.1. *(See [5, Theorem 1.4].) Let $\{u_n\}$ be a solution sequence to (1) for $\lambda = \lambda_n$ with $\lambda_n \to 16\pi$ such that u_n blows up at two points P_1 and P_2 on the annulus, Let $P_{1,n}$ and $P_{2,n}$ be the two local maximum points near P_1 and P_2 respectively, then $P_{1,n}$, $P_{2,n}$ and the origin form a straight line l_n and u_n is symmetric with respect to the line l_n for n large. Consequently, P_1, P_2 and the origin are located on a same line.*

The proof of Theorem 1.1 is done by the method of rotating planes, which is applicable to other kinds of nonlinear elliptic equations, see for example [12]. An analogous result for problems involving the critical Sobolev exponent was obtained in [4].

Theorem 1.1 leaves open the question of whether the blow up points P_1 and P_2 are anti-symmetric, i.e.

$$P_1 = -P_2. \tag{3}$$

In this note, by using the characterization of blowup points (2) and the explicit form of the Green function on an annulus derived by D.M. Hickey [10,11], we show (3).

Theorem 1.2. *Let $\{u_n\}$ be a sequence of solutions to (1) for $\lambda = \lambda_n$ with $\lambda_n \to 16\pi$ such that u_n blows up at two points P_1 and P_2 on the annulus, then we have $P_1 = -P_2$.*

Next we compute the value of $|P_1| = |P_2|$.

Theorem 1.3. *Define $r_0 = |P_1| = |P_2|$ where $P_1, P_2 \in A = \{a < |x| < a\}$ are two blowup points. Then r_0 is the unique solution of the equation*

$$2\frac{\log(r/b)}{\log(a/b)} - \frac{1}{2} = \sum_{m=1}^{\infty} \frac{1}{b^{2m} - a^{2m}} \left(r^{2m} - (ab)^{2m} r^{-2m}\right)\left((-1)^m + 1\right) \tag{4}$$

for $r \in (a, b)$.

The explicit form of the Dirichlet Green function on a two-dimensional annulus can be seen in several papers from the literature—see, for example, [6,1,2]. Most of them use the Weierstrass doubly periodic functions. We find that the Fourier expansion of the Green function is convenient to our analysis.

2. Proof of Theorem 1.2

Let $A = \{x \in \mathbb{R}^2 \mid a < |x| < b\}$ be a two-dimensional annulus. Then the Green function on A is explicitly written as follows.

Proposition 2.1 *(Hickey's formula). (See [10].) Let $G_A = G_A(x, y)$ be the Green function on A with pole $y \in A$: $-\Delta_x G_A(x, y) = 2\pi \delta_y$ in A, $G_A(x, y)|_{x \in \partial A} = 0$. Then we have*

$$G_A(x, y) = -\log|x - y| + A_0(y) + B_0(y)\log|x| - \sum_{m=1}^{\infty} \frac{1}{m}\big(A_m(y)|x|^m + B_m(y)|x|^{-m}\big)\cos m(\theta - \theta_y), \tag{5}$$

where $x = (x_1, x_2) = (|x|\cos\theta, |x|\sin\theta)$, $y = (|y|\cos\theta_y, |y|\sin\theta_y)$, and

$$A_0(y) = \log b \frac{\log(a/|y|)}{\log(a/b)}, \qquad B_0(y) = \frac{\log(|y|/b)}{\log(a/b)},$$
$$A_m(y) = \frac{|y|^m - (\frac{a^2}{|y|})^m}{b^{2m} - a^{2m}}, \qquad B_m(y) = \frac{a^{2m}((\frac{b^2}{|y|})^m - |y|^m)}{b^{2m} - a^{2m}}. \tag{6}$$

As a corollary, we have:

Corollary 2.1. *The Robin function on the annulus $A = \{a < |x| < b\} \subset \mathbb{R}^2$ is*

$$R_A(y) = -\frac{(\log|y| - \log b)^2}{\log(a/b)} - \log b + \sum_{m=1}^{\infty} \frac{1}{m}\frac{1}{b^{2m} - a^{2m}}\big(|y|^{2m} - 2a^{2m} + (ab)^{2m}|y|^{-2m}\big). \tag{7}$$

Note that R_A is a radial function on A, as it was stated in [5, Lemma 3.3].

Also using the fact $\nabla_x = \frac{x}{r}\frac{\partial}{\partial r} + \frac{x^\perp}{r^2}\frac{\partial}{\partial\theta}$ where $r = |x|$, $x^\perp = (-x_2, x_1)$ for $x = (x_1, x_2)$, we obtain the formula for the gradients of G_A and R_A as follows:

Corollary 2.2. *We have*

$$\begin{aligned}\nabla_x G_A(x, y) = &-\frac{(x - y)}{|x - y|^2} + B_0(y)\frac{x}{|x|^2} - \frac{x}{|x|}\sum_{m=1}^{\infty}\big(A_m(y)|x|^{m-1} - B_m(y)|x|^{-m-1}\big)\cos m(\theta - \theta_y) \\ &+ \frac{x^\perp}{|x|^2}\sum_{m=1}^{\infty}\big(A_m(y)|x|^m + B_m(y)|x|^{-m}\big)\sin m(\theta - \theta_y),\end{aligned} \tag{8}$$

and

$$\frac{1}{2}\nabla R_A(y) = -\frac{\log(|y|/b)}{\log(a/b)}\frac{y}{|y|^2} + \sum_{m=1}^{\infty}\frac{1}{b^{2m} - a^{2m}}\big(|y|^{2m-1} - (ab)^{2m}|y|^{-2m-1}\big)\frac{y}{|y|}. \tag{9}$$

Now, we prove Theorem 1.2 by direct calculations.

Proof of Theorem 1.2. Let $P_1, P_2 \in A$, $P_1 \neq P_2$ be two blowup points for a blowing-up solution sequence $\{u_n\}$ to (1). Since Theorem 1.1 holds, the only thing we have to prove Theorem 1.2 is that $|P_1| = |P_2|$. For that purpose, we will exploit the characterization of blowup points (2). In this case, it reads that

$$\frac{1}{2}\nabla R_A(P_1) = \nabla_x G_A(P_1, P_2), \qquad \frac{1}{2}\nabla R_A(P_2) = \nabla_x G_A(P_2, P_1), \tag{10}$$

which implies

$$\frac{1}{2}\nabla R_A(P_1)\cdot P_1 = \nabla_x G_A(P_1, P_2)\cdot P_1, \qquad \frac{1}{2}\nabla R_A(P_2)\cdot P_2 = \nabla_x G_A(P_2, P_1)\cdot P_2. \tag{11}$$

By using the formulae (8), (9), we can write Eqs. (11) as

$$\begin{aligned}&-B_0(P_1) + \sum_{m=1}^{\infty}\frac{1}{b^{2m} - a^{2m}}\big(|P_1|^{2m} - (ab)^{2m}|P_1|^{-2m}\big) \\ &= -\frac{(P_1 - P_2)\cdot P_1}{|P_1 - P_2|^2} + B_0(P_2) - \sum_{m=1}^{\infty}\big(A_m(P_2)|P_1|^m - B_m(P_2)|P_1|^{-m}\big)\cos m(\theta_{P_1} - \theta_{P_2}),\end{aligned} \tag{12}$$

and

$$\begin{aligned}
&-B_0(P_2)+\sum_{m=1}^{\infty}\frac{1}{b^{2m}-a^{2m}}\left(|P_2|^{2m}-(ab)^{2m}|P_2|^{-2m}\right)\\
&\quad=-\frac{(P_2-P_1)\cdot P_2}{|P_2-P_1|^2}+B_0(P_1)-\sum_{m=1}^{\infty}\left(A_m(P_1)|P_2|^m-B_m(P_1)|P_2|^{-m}\right)\cos m(\theta_{P_1}-\theta_{P_2}),
\end{aligned}\tag{13}$$

where $P_1=(|P_1|\cos\theta_{P_1},|P_1|\sin\theta_{P_1})$, $P_2=(|P_2|\cos\theta_{P_2},|P_2|\sin\theta_{P_2})$ in polar coordinates. Inserting (6), we have

$$\begin{aligned}
&A_m(P_2)|P_1|^m-B_m(P_2)|P_1|^{-m}\\
&\quad=\frac{1}{b^{2m}-a^{2m}}\left\{|P_1|^m|P_2|^m-a^{2m}|P_1|^m|P_2|^{-m}+a^{2m}|P_1|^{-m}|P_2|^m-(ab)^{2m}|P_1|^{-m}|P_2|^{-m}\right\},\\
&A_m(P_1)|P_2|^m-B_m(P_1)|P_2|^{-m}\\
&\quad=\frac{1}{b^{2m}-a^{2m}}\left\{|P_1|^m|P_2|^m-a^{2m}|P_1|^{-m}|P_2|^m+a^{2m}|P_1|^m|P_2|^{-m}-(ab)^{2m}|P_1|^{-m}|P_2|^{-m}\right\}.
\end{aligned}$$

Thus, subtracting (13) from (12), we have

$$\begin{aligned}
&\sum_{m=1}^{\infty}\frac{1}{b^{2m}-a^{2m}}\left(|P_1|^{2m}-(ab)^{2m}|P_1|^{-2m}-|P_2|^{2m}+(ab)^{2m}|P_2|^{-2m}\right)\\
&\quad=\frac{|P_2|^2-|P_1|^2}{|P_1-P_2|^2}-\sum_{m=1}^{\infty}\frac{2a^{2m}}{b^{2m}-a^{2m}}\left(|P_1|^{-m}|P_2|^m-|P_1|^m|P_2|^{-m}\right)\cos m(\theta_{P_1}-\theta_{P_2}).
\end{aligned}$$

From this, we obtain

$$\frac{|P_2|^2-|P_1|^2}{|P_2-P_1|^2}=\sum_{m=1}^{\infty}\frac{|P_1|^{2m}-|P_2|^{2m}}{b^{2m}-a^{2m}}\left\{1+\frac{(ab)^{2m}}{|P_1|^{2m}|P_2|^{2m}}-\frac{2a^{2m}}{|P_1|^m|P_2|^m}\cos m(\theta_{P_1}-\theta_{P_2})\right\}.\tag{14}$$

Concerning the RHS of (14), we see

$$\begin{aligned}
&\left\{1+\frac{(ab)^{2m}}{|P_1|^{2m}|P_2|^{2m}}-\frac{2a^{2m}}{|P_1|^m|P_2|^m}\cos m(\theta_{P_1}-\theta_{P_2})\right\}\\
&\quad\geq 1+\frac{(ab)^{2m}}{|P_1|^{2m}|P_2|^{2m}}-\frac{2a^mb^m}{|P_1|^m|P_2|^m}=\left(1-\frac{(ab)^m}{|P_1|^m|P_2|^m}\right)^2\geq 0,
\end{aligned}$$

since $a<b$. Thus, if $|P_1|>|P_2|$, LHS of (14) <0, while RHS of (14) ≥ 0, which is a contradiction. The case of $|P_1|<|P_2|$ leads to the same contradiction. This implies that $|P_1|=|P_2|$ must hold, which ends the proof of Theorem 1.2. □

Now we compute the value of $|P_1|=|P_2|$.

Proof of Theorem 1.3. By inserting $P_2=-P_1$ into the first equation of (10) and using (8), (9), we have

$$\begin{aligned}
&-\frac{\log(|P_1|/b)}{\log(a/b)}\frac{P_1}{|P_1|^2}+\frac{P_1}{|P_1|^2}\sum_{m=1}^{\infty}\frac{1}{b^{2m}-a^{2m}}\left(|P_1|^{2m}-(ab)^{2m}|P_1|^{-2m}\right)\\
&\quad=-\frac{1}{2}\frac{P_1}{|P_1|^2}+\frac{\log(|P_1|/b)}{\log(a/b)}\frac{P_1}{|P_1|^2}-\frac{P_1}{|P_1|^2}\sum_{m=1}^{\infty}\frac{(-1)^m}{b^{2m}-a^{2m}}\left(|P_1|^{2m}-(ab)^{2m}|P_1|^{-2m}\right),
\end{aligned}$$

which in turn implies

$$2\frac{\log(|P_1|/b)}{\log(a/b)}-\frac{1}{2}=\sum_{m=1}^{\infty}\frac{1}{b^{2m}-a^{2m}}\left(|P_1|^{2m}-(ab)^{2m}|P_1|^{-2m}\right)\left\{(-1)^m+1\right\}\tag{15}$$

since $P_1\neq 0$. Let $f(r)=2\frac{\log(r/b)}{\log(a/b)}-\frac{1}{2}$ for $a<r<b$. f is a monotonically decreasing function with $f(a+0)=\frac{3}{2}$, $f(b-0)=-\frac{1}{2}$, and having a unique zero at $r=b^{3/4}a^{1/4}$. Also define

$$g(r)=\sum_{m=1}^{\infty}\frac{1}{b^{2m}-a^{2m}}\left(r^{2m}-(ab)^{2m}r^{-2m}\right)\left\{(-1)^m+1\right\}.$$

Since $(-1)^m + 1 \geq 0$ for any $m \in \mathbb{N}$, we see g is monotonically increasing with respect to r and

$$\lim_{r \downarrow a} g(r) = \sum_{m=1}^{\infty} \frac{1}{b^{2m} - a^{2m}} \left(a^{2m} - b^{2m}\right)\left\{(-1)^m + 1\right\} = -\infty,$$

$$\lim_{r \uparrow b} g(r) = \sum_{m=1}^{\infty} \frac{1}{b^{2m} - a^{2m}} \left(b^{2m} - a^{2m}\right)\left\{(-1)^m + 1\right\} = +\infty,$$

with having unique zero $r = \sqrt{ab}$. Thus we have the unique r_0, $\sqrt{ab} < r_0 < b^{3/4}a^{1/4}$ such that $f(r_0) = g(r_0)$ by the Intermediate Value Theorem for continuous functions. $\square$

Remark 1. By the proof of the last theorem it follows that $\sqrt{ab} < r_0 < b^{3/4}a^{1/4}$.

Remark 2. It is interesting to know what will happen when the number of blowup points is three or more: we conjecture that if we have m blowup points on the two-dimensional annulus, then they must be located on the vertices of a regular m-polygon. The verification of this seems difficult.

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