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# Global exact controllability of 1D Schrödinger equations with a polarizability term

*Contrôle exact global d'équations de Schrödinger 1D avec un terme de polarisabilité*

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ABSTRACT

We consider a quantum particle in a 1D interval submitted to a potential. The evolution of this particle is controlled using an external electric field. Taking into account the so-called polarizability term in the model (quadratic with respect to the control), we prove global exact controllability in a suitable space for arbitrary potential and arbitrary dipole moment. This term is relevant both from the mathematical and physical points of view. The proof uses tools from the bilinear setting and a perturbation argument.



RÉSUMÉ

On considère une particule quantique dans un intervalle 1D, soumise à un potentiel. L'évolution de cette particule est contrôlée par un champ électrique extérieur. En prenant en compte dans le modèle le terme dit de polarisabilité (quadratique par rapport au contrôle), on prouve la contrôlabilité exacte globale dans un espace approprié pour des potentiels et des moments dipolaires arbitraires. Ce terme est intéressant à la fois d'un point de vue mathématique et physique. La preuve utilise des outils issus du cadre bilinéaire et un argument de perturbation.



## Version française abrégée

On considère une particule quantique unidimensionnelle soumise à l'action d'un potentiel $V$. La particule est représentée par sa fonction d'onde $\psi$, dont l'évolution est contrôlée par un champ électrique extérieur, d'amplitude réelle $u$. En notant $\mu_1$ le moment dipolaire et $\mu_2$ le moment de polarisabilité, l'évolution de la fonction d'onde est donnée par le système de Schrödinger avec polarisabilité :

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$$\begin{cases} \mathrm{i}\partial_t\psi = \left(-\partial^2_{xx} + V(x)\right)\psi - u(t)\mu_1(x)\psi - u(t)^2\mu_2(x)\psi, & (t,x)\in(0,T)\times(0,1), \\ \psi(t,0)=\psi(t,1)=0, \\ \psi(0,x)=\psi_0(x). \end{cases} \tag{1}$$

Si la prise en compte du terme de polarisabilité est intéressante du point de vue physique (par exemple dans le cas de contrôles de fortes amplitudes [8]), du point de vue mathématique, ce terme a permis de montrer la contrôlabilité dans des cas où le moment dipolaire est insuffisant pour conclure (voir par exemple [7,11,4]).

Pour $V\in L^2((0,1),\mathbb{R})$, on note $\lambda_{k,V}$ et $\varphi_{k,V}$ les valeurs propres (en ordre croissant) et vecteurs propres de l'opérateur $A_V$ défini sur le domaine $D(A_V):=H^2\cap H^1_0((0,1),\mathbb{C})$ par $A_V\psi:=(-\partial^2_{xx}+V(x))\psi$. On définit les états propres par $\Phi_{k,V}(t,x):=\mathrm{e}^{-\mathrm{i}\lambda_{k,V}t}\varphi_{k,V}(x)$, $(t,x)\in[0,+\infty)\times(0,1)$, $k\in\mathbb{N}^*$. Pour $s>0$, l'espace $H^s_{(V)}:=D(A_V^{s/2})$ est muni de la norme $\|\psi\|_{H^s_{(V)}}:=(\sum_{k=1}^{+\infty}|k^s\langle\psi,\varphi_{k,V}\rangle|^2)^{\frac12}$. On note $\mathcal{S}$ la sphère unité de $L^2((0,1),\mathbb{C})$.

Dans le cadre bilinéaire (c'est-à-dire pour le sytème (1) avec $\mu_2=0$), en combinant les résultats de contrôle exact local, dans $H^3_{(0)}$, autour de $\Phi_{1,0}$ de Beauchard et Laurent [2] et la contrôlabilité approchée de $\varphi_{1,0}$ dans $H^3$ du second auteur [13], on obtient la contrôlabilité exacte globale dans $\mathcal{S}\cap H^{3+\epsilon}_{(0)}$ pour $V=0$ sous des hypothèses favorables sur $\mu_1$. Ces deux résultats sont principalement basés sur l'étude de linéarisés du système au voisinage de trajectoires associées au contrôle nul. En utilisant le fait que (au moins formellement) le système (1) avec $\mu_2\in L^2((0,1),\mathbb{R})$ quelconque a le même linéarisé au voisinage de telles trajectoires que dans le cas bilinéaire $\mu_2=0$, conjointement à un argument de perturbation utilisé par les auteurs dans [12] dans le cadre du contrôle simultané de systèmes bilinéaires, on prouve le résultat suivant.

**Théorème 0.1.** *Pour tout $V,\mu_1\in H^6((0,1),\mathbb{R})$ le système (1) est globalement exactement contrôlable dans $H^6_{(V)}$, génériquement par rapport à $\mu_2\in H^6((0,1),\mathbb{R})$.*

Par rapport au modèle bilinéaire, la prise en compte du terme de polarisabilité permet de conclure à la contrôlabilité dans des cas où la contrôlabilité était fausse ou ouverte (par exemple $V$ arbitraire et $\mu_1=0$ ou $\mu_1\notin\mathcal{Q}_V$, comme défini dans [12]).

## 1. Introduction

We consider the evolution of a 1D quantum particle given by (1). The real valued functions $V$, $\mu_1$, and $\mu_2$, respectively, the potential, the dipole moment, and the polarizability moment, are given. The control $u(t)$ is real valued. The following theorem is the main result of this paper.

**Theorem 1.1.** *For any $V,\mu_1\in H^6((0,1),\mathbb{R})$, system (1) is globally exactly controllable in $H^6_{(V)}$ generically with respect to $\mu_2\in H^6((0,1),\mathbb{R})$. More precisely, there is a residual set $\mathcal{Q}_{V,\mu_1}$ in $H^6((0,1),\mathbb{R})$ such that if $\mu_2\in\mathcal{Q}_{V,\mu_1}$, then for any $\psi_0,\psi_{\mathrm{f}}\in\mathcal{S}\cap H^6_{(V)}$, there is $T>0$ and $u\in H^1_0((0,T),\mathbb{R})$ such that the solution of (1) satisfies $\psi(T)=\psi_{\mathrm{f}}$.*

Essentially with the same proof, one can establish the same exact controllability property in the case where the term $u(t)^2\mu_2(x)\psi$ in (1) is replaced by a higher-degree term $\sum_{j=2}^m u^j\mu_j\psi$. We choose $m=2$ for the sake of simplicity of presentation.

*Review of previous results*. The controllability properties of quantum particles were first studied for the bilinear model (i.e., for (1) with $\mu_2=0$). In [2], Beauchard and Laurent proved local exact controllability in $H^3_{(0)}$ around $\Phi_{1,0}$ by studying the controllability of the linearized system around the trajectory $(u\equiv 0,\Phi_{1,0})$. The simultaneous global exact controllability of an arbitrary (finite) number of such bilinear equations was studied by the authors in [12] for arbitrary potentials. For multidimensional domains, we mention the simultaneous approximate controllability property obtained by Boscain, Caponigro, Chambrion, Mason, Sigalotti [6,5] through geometric techniques based on the exact controllability of the Galerkin approximations. The approximate controllability in Sobolev spaces towards the state $\varphi_{1,V}$ is obtained by the second author using Lyapunov techniques [13]. The first controllability results for systems having a polarizability term are established for finite-dimensional models and are due to Coron, Grigoriu, Lefter, and Turinici [15,10,7,9]. They proved exact controllability under the same assumptions as for the bilinear model. They also proved stabilization of the first eigenstate using either discontinuous feedback laws or time-oscillating periodic feedback laws in a setting where the dipole moment was not sufficient to conclude. The strategy based on time-oscillating feedback laws has been extended to the infinite dimensional model (1) by the first author [11]. Finally, geometric technics were applied to the polarizability system by Boussaid, Caponigro, Chambrion in [4], leading to global approximate controllability.

*Structure of the article*. First, we prove in Section 2 approximate controllability towards the ground state, adapting Lyapunov arguments from the bilinear setting. Still using tools from the bilinear setting, we prove in Section 3 local exact controllability around the ground state, in $H^5_{(V)}$, with controls in $H^1_0((0,T),\mathbb{R})$. Gathering these results, we get global exact controllability under favorable hypotheses on $V$ and $\mu_1$ in Section 4. Then, we conclude the proof using a perturbation argument.

## 2. Approximate controllability towards the ground state

From the arguments of the proofs of [14, Proposition 3.1] and [2, Proposition 5], since $H_0^1((0,T),\mathbb{R})$ is an algebra, it follows that for any $V,\mu_1,\mu_2 \in H^5((0,1),\mathbb{R})$, $T>0$, $\psi_0 \in H^5_{(V)}$, and $u \in H_0^1((0,T),\mathbb{R})$, system (1) has a unique weak solution $\psi \in C([0,T],H^5)\cap C^1([0,T],H^3_{(V)})$. Furthermore, the mapping which sends $(\psi_0,u)$ to the solution $\psi$ is $C^1$. As $u(T)=0$, it comes that $\psi(T)\in H^5_{(V)}$.

Proceeding as in [1, Proposition 49], as $\mu_1,\mu_2 \in H^6((0,1),\mathbb{R})$, when $\psi_0 \in H^6_{(V-u(0)\mu_1-u(0)^2\mu_2)}$, $u \in C^2([0,T],\mathbb{R})$, and $\dot u(0)=0$, we obtain that the solution $\psi$ belongs to $C([0,T],H^6)$. Moreover, if $\dot u(T)=0$, then $\psi(T,\psi_0,u)\in H^6_{(V-u(T)\mu_1-u(T)^2\mu_2)}$. Let us introduce the following Lyapunov function

$$\mathcal{L}(z) := \gamma \left\| \left(-\partial_{xx}^2+V\right)^3 \mathcal{P}z\right\|_{L^2}^2 + 1 - \left|\langle z,\varphi_{1,V}\rangle\right|^2, \quad z\in \mathcal{S}\cap H^6_{(V)}, \tag{2}$$

where $\mathcal{P}$ is the orthogonal projection in $L^2$ onto the closure of the vector space spanned by $\{\varphi_{k,V};\ k\geqslant 2\}$ and $\gamma>0$ is a constant which will be clarified later on. This Lyapunov function has already been used in the bilinear setting by the second author and Beauchard in [13,3], and adapted to study simultaneous controllability in [12] by the authors. We assume that the functions $V,\mu_1 \in H^6((0,1),\mathbb{R})$ are such that

**(C₁)** $\langle \mu_1\varphi_{1,V},\varphi_{k,V}\rangle \neq 0$, for all $k\in\mathbb{N}^*$,
**(C₂)** $\lambda_{1,V}-\lambda_{j,V}\neq \lambda_{p,V}-\lambda_{q,V}$, for all $j,p,q\geqslant 1$ and $j\neq 1$.

**Theorem 2.1.** *Let $V,\mu_1,\mu_2\in H^6((0,1),\mathbb{R})$ be such that Conditions* **(C₁)** *and* **(C₂)** *are satisfied. For any $\psi_0\in\mathcal{S}\cap H^6_{(V)}$ satisfying $\langle\psi_0,\varphi_{1,V}\rangle\neq 0$ and for any $\varepsilon>0$, there are $T>0$ and $u\in C_0^2((0,T),\mathbb{R})$ such that $\|\psi(T,\psi_0,u)-\varphi_{1,V}\|_{H^5}<\varepsilon$.*

**Proof.** For $\mu_2=0$, the proof is given in [13, Theorem 2.3]. The adaptation to $\mu_2\in H^6((0,1),\mathbb{R})$ is straightforward; we only recall the scheme of the proof. Since for every $\psi_0\in H^6_{(V)}$, the linearization of (1) around the trajectory $\psi(\cdot,\psi_0,0)$ is the same for $\mu_2=0$ or any other $\mu_2\in H^6((0,1),\mathbb{R})$, we deduce from [13, Proposition 2.6] the following lemma (the set $J$ used in the original proposition is empty in this case).

**Lemma 2.2.** *Let $V,\mu_1,\mu_2\in H^6((0,1),\mathbb{R})$ be such that Conditions* **(C₁)** *and* **(C₂)** *are satisfied. For any $\psi_0\in\mathcal{S}\cap H^6_{(V)}$ satisfying $\langle\psi_0,\varphi_{1,V}\rangle\neq 0$ and $\mathcal{L}(\psi_0)>0$, there exist a time $T>0$ and a control $u\in C_0^2((0,T),\mathbb{R})$ such that $\mathcal{L}(\psi(T,\psi_0,u))<\mathcal{L}(\psi_0)$.*

Let us take any $\psi_0\in\mathcal{S}\cap H^6_{(V)}$ satisfying $\langle\psi_0,\varphi_{1,V}\rangle\neq 0$ and let us choose the constant $\gamma>0$ in (2) such that $\mathcal{L}(\psi_0)<1$. If $\mathcal{L}(\psi_0)>0$, we define

$$\mathcal{K} := \left\{\psi_{\mathrm{f}}\in H^6_{(V)};\ \psi(T_n,\psi_0,u_n)\underset{n\to\infty}{\longrightarrow}\psi_{\mathrm{f}},\ \text{in } H^5 \text{ where } T_n>0,\ u_n\in C_0^2\big((0,T_n),\mathbb{R}\big)\right\}.$$

Using a minimizing sequence, we get that the infimum of $\mathcal{L}$ on $\mathcal{K}$ is attained, i.e., there is $e\in\mathcal{K}$ such that $\mathcal{L}(e)=\inf_{\psi\in\mathcal{K}}\mathcal{L}(\psi)$. This gives that $\mathcal{L}(e)\leqslant\mathcal{L}(\psi_0)<1$, hence $\langle e,\varphi_{1,V}\rangle\neq 0$. Using Lemma 2.2, it comes that if $\mathcal{L}(e)>0$, then there are $T>0$ and $u\in C_0^2((0,T),\mathbb{R})$ such that $\mathcal{L}(\psi(T,e,u))<\mathcal{L}(e)$. As $\psi(T,e,u)\in\mathcal{K}$, this contradicts the definition of $e$. Then, $\mathcal{L}(e)=0$. This leads to $\varphi_{1,V}\in\mathcal{K}$ and concludes the proof of Theorem 2.1. □

## 3. Local exact controllability around the ground state

In this section, we prove local exact controllability around the ground state $\varphi_{1,V}$ in $H^5_{(V)}$ with controls in $H_0^1((0,T),\mathbb{R})$. We assume that the functions $V,\mu_1\in H^5((0,1),\mathbb{R})$ satisfy:

**(C₃)** there exists $C>0$ such that $|\langle\mu_1\varphi_{1,V},\varphi_{k,V}\rangle|\geqslant \frac{C}{k^3}$, for all $k\in\mathbb{N}^*$.

**Theorem 3.1.** *Let $V,\mu_1,\mu_2\in H^5((0,1),\mathbb{R})$ be such that Condition* **(C₃)** *is satisfied. Let $T>0$. There exist $\delta>0$ and a $C^1$ map $\Gamma:\mathcal{O}_T\longrightarrow H_0^1((0,T),\mathbb{R})$, where $\mathcal{O}_T:=\{\psi_{\mathrm{f}}\in\mathcal{S}\cap H^5_{(V)};\ \|\psi_{\mathrm{f}}-\Phi_{1,V}(T)\|_{H^5}<\delta\}$, such that $\Gamma(\Phi_{1,V}(T))=0$, and for any $\psi_{\mathrm{f}}\in\mathcal{O}_T$, the solution of (1) with initial condition $\psi_0=\varphi_{1,V}$ and control $u=\Gamma(\psi_{\mathrm{f}})$ satisfies $\psi(T)=\psi_{\mathrm{f}}$.*

**Proof.** In the case where $\mu_2=0$ and $V=0$, the proof is exactly the one of [2, Theorem 2]. Let $\mathcal{H}:=\{\psi\in H^5_{(V)};\ \Re(\langle\psi,\varphi_{1,V}\rangle)=0\}$, and let $\mathcal{P}_{\mathcal{H}}$ be the orthogonal projection in $L^2((0,1),\mathbb{C})$ onto $\mathcal{H}$. Then the end-point map $\Theta_T: u\in H_0^1((0,T),\ \mathbb{R})\mapsto \mathcal{P}_{\mathcal{H}}(\psi(T,\varphi_{1,V},u))\in\mathcal{H}$, is $C^1$ and its differential at 0 is given by $\mathrm{d}\Theta_T(0)v=\Psi(T)$, where $\Psi$ is the solution of

$$\mathrm{i}\partial_t\Psi=\left(-\partial_{xx}^2+V(x)\right)\Psi - v(t)\mu_1(x)\Phi_{1,V}, \quad (t,x)\in(0,T)\times(0,1) \tag{3}$$

with homogeneous Dirichlet boundary conditions and $\Psi(0,x)=0$. Rewriting this in the Duhamel form, we get

$$\Psi(T)=\mathrm{i}\sum_{k=1}^{+\infty}\langle\mu_1\varphi_{1,V},\varphi_{k,V}\rangle\int_0^T v(t)\mathrm{e}^{\mathrm{i}(\lambda_{k,V}-\lambda_{1,V})t}\,\mathrm{d}t\,\Phi_{k,V}(T).$$

Using Condition $(\mathbf{C_3})$, the asymptotics of eigenvalues $\lambda_{k,V}$, and [2, Corollary 2], we get the existence of a continuous linear map $\mathcal{M}:\mathcal{H}\mapsto L^2((0,T),\mathbb{R})$ such that for any $\Psi_\mathrm{f}\in\mathcal{H}$, the function $w:=\mathcal{M}(\Psi_\mathrm{f})$ solves the following moment problem:

$$\begin{cases}\displaystyle\int_0^T w(t)\,\mathrm{d}t=0, \qquad \int_0^T w(t)(T-t)\,\mathrm{d}t=\frac{1}{\langle\mu_1\varphi_{1,V},\varphi_{1,V}\rangle}\big\langle\Psi_\mathrm{f},\Phi_{1,V}(T)\big\rangle,\\ \displaystyle\int_0^T w(t)\,\mathrm{e}^{\mathrm{i}(\lambda_{k,V}-\lambda_{1,V})t}\,\mathrm{d}t=\frac{\lambda_{1,V}-\lambda_{k,V}}{\langle\mu_1\varphi_{1,V},\varphi_{k,V}\rangle}\big\langle\Psi_\mathrm{f},\Phi_{k,V}(T)\big\rangle,\quad \forall k\geqslant 2.\end{cases}$$

Then the mapping $\Psi_\mathrm{f}\in\mathcal{H}\mapsto\int_0^t w(\tau)\,\mathrm{d}\tau\in H_0^1((0,T),\mathbb{R})$, is a continuous right inverse for the differential $\mathrm{d}\Theta_T(0)$. Finally, applying the inverse mapping theorem to $\Theta_T$ at $u=0$, using the conservation of the $L^2$ norm and the hypothesis that $\psi_\mathrm{f}\in\mathcal{S}$, we complete the proof of Theorem 3.1. $\square$

**Remark 1.** Let us notice that the linearized system (3) is controllable in $H^3_{(V)}$ with controls in $L^2((0,T),\mathbb{R})$, but we cannot conclude to controllability for (1), since we do not know if the latter is well posed in $H^3_{(V)}$ for $u\in L^2((0,T),\mathbb{R})$. Indeed, in that case $u^2$ will be in $L^1((0,T),\mathbb{R})$, which does not allow to apply the results of [2].

## 4. Global exact controllability

Combining the properties of global approximate controllability obtained in Theorem 2.1 and local exact controllability obtained in Theorem 3.1, we obtain global exact controllability of (1) in $H^6_{(V)}$ under favorable hypotheses on $V$ and $\mu_1$.

**Theorem 4.1.** *Let $V,\mu_1,\mu_2\in H^6((0,1),\mathbb{R})$ be such that Conditions $(\mathbf{C_2})$ and $(\mathbf{C_3})$ are satisfied. For any $\psi_0,\psi_\mathrm{f}\in\mathcal{S}\cap H^6_{(V)}$, there is a time $T>0$ and a control $u\in H_0^1((0,T),\mathbb{R})$ such that the associated solution of (1) satisfies $\psi(T)=\psi_\mathrm{f}$.*

**Proof.** *First step*. Let $\psi_0,\psi_\mathrm{f}\in\mathcal{S}\cap H^6_{(V)}$ be such that $\langle\psi_0,\varphi_{1,V}\rangle\neq 0$ and $\langle\psi_\mathrm{f},\varphi_{1,V}\rangle\neq 0$. Let $T_*>0$ be such that $\Phi_{1,V}(T_*)=\varphi_{1,V}$ and let $\delta>0$ be the radius of local exact controllability in $H^5_{(V)}$ in time $T_*$ given by Theorem 3.1. Theorem 2.1 implies the existence of times $T_0,T_\mathrm{f}>0$ and controls $u_0\in C_0^2((0,T_0),\mathbb{R})$, $u_\mathrm{f}\in C_0^2((0,T_\mathrm{f}),\mathbb{R})$ such that

$$\big\|\psi(T_0,\psi_0,u_0)-\varphi_{1,V}\big\|_{H^5}+\big\|\psi(T_\mathrm{f},\overline{\psi_\mathrm{f}},u_\mathrm{f})-\varphi_{1,V}\big\|_{H^5}<\delta.$$

By Theorem 3.1, there exists $u_*\in H_0^1((0,T_*),\mathbb{R})$ such that $\psi(T_*,\varphi_{1,V},u_*)=\overline{\psi(T_0,\psi_0,u_0)}$. Time reversibility property of (1) implies that, if we define $u(t)=u_0(t)$ for $t\in[0,T_0]$ and $u(t+T_0)=u_*(T_*-t)$ for $t\in[0,T_*]$, then $u\in H_0^1((0,T_0+T_*),\mathbb{R})$ and $\psi(T_0+T_*,\psi_0,u)=\varphi_{1,V}$. The same arguments lead to the existence of $\tilde{u}\in H_0^1((0,T_\mathrm{f}+T_*),\mathbb{R})$ such that $\psi(T_\mathrm{f}+T_*,\overline{\psi_\mathrm{f}},\tilde{u})=\varphi_{1,V}$. Taking $T:=T_0+T_\mathrm{f}+2T_*$ and again applying the time reversibility, we find $u\in H_0^1((0,T),\mathbb{R})$ satisfying $\psi(T,\psi_0,u)=\psi_\mathrm{f}$.

*Second step*. It only remains to remove the hypotheses $\langle\psi_0,\varphi_{1,V}\rangle\neq 0$ and $\langle\psi_\mathrm{f},\varphi_{1,V}\rangle\neq 0$. Using time reversibility, it is sufficient to prove that for any $\psi_0\in\mathcal{S}$, there are $T>0$ and $u\in C_0^2((0,T),\mathbb{R})$ such that $\langle\psi(T,\psi_0,u),\varphi_{1,V}\rangle\neq 0$. Let $\widehat{\psi}_0\in\mathcal{S}\cap H^6_{(V)}$ be such that $\langle\widehat{\psi}_0,\varphi_{1,V}\rangle\neq 0$ and $\|\psi_0-\widehat{\psi}_0\|_{L^2}<\sqrt{2}$. From the first step, we get the existence of $T>0$ and $\hat{u}\in H_0^1((0,T),\mathbb{R})$ such that $\psi(T,\widehat{\psi}_0,\hat{u})=\varphi_{1,V}$. Then, the conservation of the $L^2$ norm implies:

$$\big\|\psi(T,\psi_0,\hat{u})-\varphi_{1,V}\big\|_{L^2}=\|\psi_0-\widehat{\psi}_0\|_{L^2}<\sqrt{2}.$$

Then, for $u\in C_0^2((0,T),\mathbb{R})$ sufficiently close to $\hat{u}$ in $L^2((0,T),\mathbb{R})$, we get $\langle\psi(T,\psi_0,u),\varphi_{1,V}\rangle\neq 0$, which is sufficient to apply the first step. This completes the proof of Theorem 4.1. $\square$

Finally, adapting a perturbation argument from [12] and applying Theorem 4.1, we prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $V,\mu_1\in H^6((0,1),\mathbb{R})$ and let $\mathcal{Q}_{V,\mu_1}$ be the set of functions $\mu_2\in H^6((0,1),\mathbb{R})$ such that Conditions $(\mathbf{C_2})$ and $(\mathbf{C_3})$ are satisfied with the functions $V$ and $\mu_1$ replaced, respectively, by $V-2\mu_1-4\mu_2$ and $\mu_1+4\mu_2$, i.e., $\mathcal{Q}_{V,\mu_1}:=\{\mu_2\in H^6((0,1),\mathbb{R});\ \text{Conditions } (\mathbf{C_2'}) \text{ and } (\mathbf{C_3'}) \text{ hold}\}$, where

**($C_2'$)** $\lambda_{1,V-2\mu_1-4\mu_2} - \lambda_{j,V-2\mu_1-4\mu_2} \neq \lambda_{p,V-2\mu_1-4\mu_2} - \lambda_{q,V-2\mu_1-4\mu_2}$, for $j, p, q \geqslant 1$ and $j \neq 1$,
**($C_3'$)** there exists $C > 0$ such that $|\langle(\mu_1 + 4\mu_2)\varphi_{1,V-2\mu_1-4\mu_2}, \varphi_{k,V-2\mu_1-4\mu_2}\rangle| \geqslant \frac{C}{k^3}$, for every $k \in \mathbb{N}^*$.

*First step: global exact controllability when $\mu_2 \in \mathcal{Q}_{V,\mu_1}$*. Let us consider the equation:

$$\mathrm{i}\partial_t\psi = \big(-\partial_{xx}^2 + V(x) - 2\mu_1(x) - 4\mu_2(x)\big)\psi - \tilde{u}(t)(\mu_1 + 4\mu_2)(x)\psi - \tilde{u}(t)^2\mu_2(x)\psi \tag{4}$$

for $(t,x) \in (0,T) \times (0,1)$ and homogeneous Dirichlet boundary conditions. We denote by $\widetilde{\psi}(T,\psi_0,u)$ its solution at time $T$. Then

$$\widetilde{\psi}(t,\psi_0,u) = \psi(t,\psi_0,u+2) \quad \text{for } t \in [0,T]. \tag{5}$$

Let $\psi_0, \psi_\mathrm{f} \in \mathcal{S} \cap H^6_{(V)}$ and let $u_1 \in C^2([0,1],\mathbb{R})$ be such that $u_1(0) = \dot{u}_1(0) = \dot{u}_1(1) = 0$ and $u_1(1) = 2$. Then $\widetilde{\psi}_0 := \psi(1,\psi_0,u_1) \in \mathcal{S} \cap H^6_{(V-2\mu_1-4\mu_2)}$ and $\overline{\widetilde{\psi}_\mathrm{f}} := \psi(1,\overline{\psi_\mathrm{f}},u_1) \in \mathcal{S} \cap H^6_{(V-2\mu_1-4\mu_2)}$. As $\mu_2 \in \mathcal{Q}_{V,\mu_1}$, Theorem 4.1 implies the existence of $\tilde{T} > 0$ and $\tilde{u} \in H^1_0((0,\tilde{T}),\mathbb{R})$ such that $\widetilde{\psi}(\tilde{T},\widetilde{\psi}_0,\tilde{u}) = \widetilde{\psi}_\mathrm{f}$. Let $T = 2 + \tilde{T}$ and $u(t) = u_1(t)$ for $t \in [0,1]$, $u(t) = \tilde{u}(t-1) + 2$ for $t \in [1, \tilde{T}+1]$, and $u(t) = u_1(1-(t-1-\tilde{T}))$ for $t \in [\tilde{T}+1, T]$. Then, time reversibility of (1) and (5) implies $\psi(T,\psi_0,u) = \psi_\mathrm{f}$ with $u \in H^1_0((0,T),\mathbb{R})$.

*Second step: genericity*. We conclude the proof of Theorem 1.1 by showing that $\mathcal{Q}_{V,\mu_1}$ is residual in $H^6((0,1),\mathbb{R})$. For any $W \in H^6((0,1),\mathbb{R})$, let $\mathcal{Q}_W$ be the set of functions $\mu \in H^6((0,1),\mathbb{R})$ such that

- $\lambda_{1,W+\mu} - \lambda_{j,W+\mu} \neq \lambda_{p,W+\mu} - \lambda_{q,W+\mu}$, for $j, p, q \geqslant 1$ and $j \neq 1$,
- there exists $C > 0$ such that $|\langle\mu\varphi_{1,W+\mu}, \varphi_{k,W+\mu}\rangle| \geqslant \frac{C}{k^3}$, for every $k \in \mathbb{N}^*$.

By [12, Lemma 5.3] with $s = 6$, the set $\mathcal{Q}_W$ is residual in $H^6((0,1),\mathbb{R})$. Let $W := V - \mu_1 \in H^6((0,1),\mathbb{R})$. For any $\mu \in \mathcal{Q}_W$, if we set $\mu_2 := -\frac{1}{4}(\mu_1 + \mu)$, then $\mu_2 \in \mathcal{Q}_{V,\mu_1}$. This ends the proof of Theorem 1.1. $\square$

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