C. R. Acad. Sci. Paris, Ser. I 352 (2014) 435–439

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Functional analysis

Sums of unitarily equivalent positive operators

Sommes d'opérateurs unitairement équivalents

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ARTICLE INFO

Article history:
Received 12 December 2013
Accepted after revision 17 March 2014
Available online 27 March 2014

Presented by Jean-Michel Bony

ABSTRACT

Some simple conditions on positive operators A and K ensure that A can be written as a series in the unitary orbit of K.

RÉSUMÉ

Des conditions simples sur les opérateurs positifs A et K assurent que A s'écrit comme une série dans l'orbite unitaire de K.

1. Sums in a unitary orbit

A natural relation on operators (bounded linear operators on a fixed separable, real or complex, infinite-dimensional Hilbert space $\mathcal{H}$) is the unitary equivalence: $A \simeq B$, meaning that $A = UBU^*$ for some unitary operator U, i.e., A and B have the same unitary orbit. For positive operators (self-adjoint operators with nonnegative spectra), we will prove the following decomposition in a unitary orbit. An operator T is called *nonsingular* if $\ker T = 0$. If $\|Th\| < \|h\|$ for all nonzero $h \in \mathcal{H}$, then T is called a *strict contraction*. The essential norm of T is denoted by $\|T\|_e$ and its essential spectrum by $\mathrm{Sp}_e(T)$.

Theorem 1.1. *Let A be a nonsingular positive operator such that $\|A\|_e \geqslant 1$ and let $K \neq 0$ be a positive strict contraction such that $0 \in \mathrm{Sp}_e(K)$. Then, there exists a decomposition*

$$A = \sum_{j=1}^{\infty} K_j$$

where $K_j \simeq K$ for each $j \in \mathbb{N}^*$.

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[1] Research supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2013-R1A1A2059872).

http://dx.doi.org/10.1016/j.crma.2014.03.012
1631-073X/

The assumption $0 \in \mathrm{Sp}_e(K)$ is necessary: otherwise, there would exist a projection E with a range of finite codimension and a scalar $s > 0$, for instance $s = \min\{t \in \mathrm{Sp}_e(K)\}/2$, such that $K \geqslant sE$, hence $\|\sum_{j=1}^m K_j\| \geqslant ms \to \infty$ as $m \to \infty$. The condition $\|A\|_e \geqslant 1$ is obviously necessary to ensure that the theorem holds for any positive strict contraction K with 0 in the essential spectrum, including the case of $\|K\|_e = 1$. The theorem claims that this condition is also sufficient.

Of course, the series in the theorem refers to the convergence in the strong operator topology; norm convergence cannot hold. Relaxing strong convergence to weak convergence would lead to the same statement since for an increasing sequence of positive operators, the three properties "weakly convergent", "strongly convergent", "bounded" are equivalent (Vigier's theorem).

The proof of the theorem relies on our recent note [1] where a similar statement is obtained for the Murray–von Neumann equivalence on positive operators: $A \sim B$ if $A = TT^*$ and $B = T^*T$ for some operator T. The relation $\sim$ is weaker than $\simeq$, though coinciding on finite-rank positive operators. Thus this work is a continuation of [1] and an underlying tool is a matrix decomposition from [2, Lemma 3.4]. As explained in [1], this tool is not available in the operator algebra setting; however, it seems natural to propose the following conjecture.

Conjecture 1.2. *Let A, K be two positive operators in a type-II_∞ or -III factor $\mathcal{M}$. If A, K meet the assumptions of the theorem, then the conclusion holds with the unitary equivalence in $\mathcal{M}$. (In a type-III factor, $\|\cdot\|_e := \|\cdot\|$, and in a type II_∞ factor, $\|\cdot\|_e$ is defined via "compact" perturbations in $\mathcal{M}$.)*

2. Main steps of the proof of Theorem 1.1

The question "Which operators are a strong sum of projections?" is natural, and a detailed study is given by Kaftal, Ng and Zhang [4], improving an earlier work of Dykema et al. [3]. Considering contractions in the same $\sim$-equivalence class rather than projections was the purpose of [1]. The main result of [1] essentially contains the following lemma, implicit in [3] in case of a projection B. A contraction B means $\|Bh\| \leqslant \|h\|$ for all $h \in \mathcal{H}$.

Lemma 2.1. *Let A be a positive operator such that $\|A\|_e > 1$ and let $B \neq 0$ be a positive contraction. Then, there exists a decomposition*

$$A = \sum_{j=1}^{\infty} B_j$$

where $B_j \sim B$ and $0 \in \mathrm{Sp}_e(B_j)$ for each $j \in \mathbb{N}^*$.

Proof. From [1], the decomposition holds with $B_j \sim B$, and it remains to check that we can assume $0 \in \mathrm{Sp}_e(B_j)$, $j \in \mathbb{N}^*$. Split A as a direct sum $A = A_1 \oplus \cdots \oplus A_k \oplus \cdots$, corresponding to a decomposition of $\mathcal{H}$ in invariant subspaces of A, such that $\|A_k\|_e = \|A\|_e$ for all $k \in \mathbb{N}^*$. So, by [1], regarding A_k as an operator on $\mathcal{H}$, we have a strong sum decomposition

$$A_k = \sum_{j=1}^{\infty} B_j^{(k)}$$

where $B_j^{(k)} \sim B$, $j, k \in \mathbb{N}^*$. As $\dim\ker B_j^{(k)} = \infty$, we have $0 \in \mathrm{Sp}_e(B_j^{(k)})$. Relabeling the double sequence $\{B_j^{(k)}\}$ as a simple sequence $\{B_j\}$ completes the proof. □

To derive Theorem 1.1 from Lemma 2.1, we need some auxiliary lemmas. The two next ones are elementary.

Lemma 2.2. *Let A be an invertible positive operator such that $\|A\|_e > 1$. Then, there exists a family of invertible positive operators $\{A_n\}_{n=1}^\infty$ such that*

$$A = \sum_{n=1}^{\infty} A_n$$

and $\|A_n\|_e > 1$ for each $n \in \mathbb{N}^*$.

Given a positive operator A and a scalar $t > 0$, the inequality $A \geqslant t$ means that $A \geqslant tI$, where I is the identity.

Proof. We need a simple fact: if A' is a positive operator on an infinite-dimensional Hilbert space $\mathcal{H}'$, $A' \geqslant s$ for some $s > 0$, then there exists a family of mutually orthogonal infinite projections $\{Q_n\}_{n=1}^\infty$ summing up to the identity on $\mathcal{H}'$ such that $A'Q_n = Q_n'A$ for each $n \in \mathbb{N}^*$ (if the spectrum of A is finite, then some projections Q_n are not spectral projections of A').

Since $\|A\|_e > 1$ and A is invertible, there exists $t > 0$ such that $\|A\|_e \geqslant 1 + 2t$ and $A \geqslant 2t$. Let P be a spectral projection of A such that $AP \geqslant (1+t)P$, $\mathrm{rank}P = \infty$. The above fact, with $A' = AP$ and $\mathcal{H}' = P(\mathcal{H})$, shows that there exists a family $\{P_n\}_{n=1}^\infty$ of mutually orthogonal infinite projections such that

$$P=\sum_{n=1}^{\infty} P_n$$

and $AP_n = P_nA$ for each $n \in \mathbb{N}^*$. Define, for all $n \in \mathbb{N}^*$, $B_n := P_n + t2^{-n}I$. Hence each B_n is invertible and satisfies $\|B_n\|_e > 1$. Furthermore,

$$\sum_{n=1}^{\infty} B_n = P + tI \leqslant A.$$

Setting, for all $n \geqslant 2$, $A_n := B_n$, and for $n = 1$,

$$A_1 = B_1 + \left(A - \sum_{n=1}^{\infty} B_n\right)$$

completes the proof. □

Lemma 2.3. *Fix $\varepsilon > 0$ and two positive operators $K \sim L$. If $0 \in \mathrm{Sp}_e(K)$ and $0 \in \mathrm{Sp}_e(L)$, then there exists a positive operator $M \simeq K$ such that $\|L - M\| < \varepsilon$.*

Proof. By assumption $L = VKV^*$, where V is a partial isometry on $\mathcal{H}$ which can be regarded as an onto isometry between the support subspaces $\mathcal{H} \ominus \ker K$ and $\mathcal{H} \ominus \ker L$.

Since 0 belongs to the essential spectra, for all $\varepsilon > 0$, the spectral subspaces of K and L corresponding to the interval $[0, \varepsilon)$, denoted by $\mathcal{S}_\varepsilon(K)$ and $\mathcal{S}_\varepsilon(L)$ respectively, have infinite dimension. We have two direct sum decompositions,

$$\mathcal{H} = \mathcal{S}_\varepsilon(K) \oplus \mathcal{S}_\varepsilon^\perp(K) = \mathcal{S}_\varepsilon(L) \oplus \mathcal{S}_\varepsilon^\perp(L)$$

and V induces an isometry from $\mathcal{S}_\varepsilon^\perp(K)$ onto $\mathcal{S}_\varepsilon^\perp(L)$. Writing $K = K_\varepsilon^- + K_\varepsilon^+$, where K_ε^- is the compression of K onto $\mathcal{S}_\varepsilon(K)$ and K_ε^+ is the compression of K onto $\mathcal{S}_\varepsilon^\perp(K)$, and similarly $L = L_\varepsilon^- + L_\varepsilon^+$, we have $L_\varepsilon^+ = VK_\varepsilon^+V^*$. We may extend the isometry $V : \mathcal{S}_\varepsilon^\perp(K) \to \mathcal{S}_\varepsilon^\perp(L)$ to a unitary U on $\mathcal{H}$, by picking any onto isometry between $\mathcal{S}_\varepsilon(K)$ and $\mathcal{S}_\varepsilon(L)$. We then have:

$$\left\|UKU^* - L\right\| = \left\|VK_\varepsilon^+V^* + UK_\varepsilon^-U^* - \left(L_\varepsilon^+ + L_\varepsilon^-\right)\right\| = \left\|UK_\varepsilon^-U^* - L_\varepsilon^-\right\| \leqslant 2\varepsilon,$$

so replacing ε by $\varepsilon/2$ and setting $M = UKU^*$ yields the statement of the lemma. □

Lemma 2.4. *Let A be an invertible positive operator such that $\|A\|_e > 1$ and let $K \neq 0$ be a positive contraction such that $0 \in \mathrm{Sp}_e(K)$. Then, there exist $\varepsilon > 0$ arbitrarily small and some operators $M_j \simeq K$, for each $j \in \mathbb{N}^*$, such that*

$$3\varepsilon \geqslant A - \sum_{j=1}^{\infty} M_j \geqslant \varepsilon.$$

Proof. Take $\varepsilon > 0$ small enough to have $A - 2\varepsilon \geqslant 0$ and $\|A - 2\varepsilon\|_e > 1$. By Lemma 2.1,

$$A - 2\varepsilon = \sum_{j=1}^{\infty} L_j \tag{2.1}$$

where $L_j \sim K$ and $0 \in \mathrm{Sp}_e(L_j)$ for each $j \in \mathbb{N}^*$. By Lemma 2.3, we have $M_j \simeq K$ such that

$$\|L_j - M_j\| < \varepsilon 2^{-j} \tag{2.2}$$

for all $j \in \mathbb{N}^*$. Combining (2.1) and (2.2) we obtain

$$A - 2\varepsilon = E + \sum_{j=1}^{\infty} M_j$$

where the norm convergent series

$$E := \sum_{j=1}^{\infty} (L_j - M_j)$$

is self-adjoint and satisfies $\|E\| < \varepsilon$. Hence

$$A-\sum_{j=1}^{\infty} M_j = 2\varepsilon + E$$

satisfies the claimed two bounds. □

Proof of Theorem 1.1. I. We first assume $\|A\|_e > 1$ and consider two cases.

(1) A is invertible. By Lemma 2.2, we have a series of positive operators

$$A=\sum_{n=1}^{\infty} A_n$$

where A_n is invertible, say $A_n \geqslant \alpha_n > 0$, and $\|A_n\|_e > 1$ for all $n \in \mathbb{N}^*$. We then build up a sequence of positive operators $\{R_n\}_{n=1}^{\infty}$ as follows.

- By Lemma 2.4 applied to A_1 with $0<\varepsilon<\min\{\alpha_1, 2^{-1}\}$, we define R_1 such as

$$A_1=\left(\sum_{j=1}^{\infty} M_j^{(1)}\right)+R_1$$

  where $3\min\{\alpha_1, 2^{-1}\} \geqslant R_1 \geqslant 0$ and $M_j^{(1)} \simeq K$ for each $j \in \mathbb{N}^*$.
- Assuming that $R_1,\dots,R_{m-1}$ have been built up, we choose R_m by applying Lemma 2.4 to $R_{m-1}+A_m$ with $0<\varepsilon<\min\{\alpha_m, 2^{-m}\}$,

$$R_{m-1}+A_m=\left(\sum_{j=1}^{\infty} M_j^{(m)}\right)+R_m$$

  where $3\min\{\alpha_m, 2^{-m}\} \geqslant R_m \geqslant 0$ and $M_j^{(m)} \simeq K$ for each $j \in \mathbb{N}^*$.

Therefore, we have:

$$\begin{aligned}
A &= A_1+A_2+\cdots+A_m+\cdots\\
&=\left(\sum_{j=1}^{\infty} M_j^{(1)}\right)+R_1+A_2+\cdots+A_m+\cdots\\
&=\left(\sum_{j=1}^{\infty} M_j^{(1)}\right)+\left(\sum_{j=1}^{\infty} M_j^{(2)}\right)+R_2+A_3+\cdots+A_m+\cdots\\
&=\left(\sum_{j=1}^{\infty} M_j^{(1)}\right)+\left(\sum_{j=1}^{\infty} M_j^{(2)}\right)+\cdots+\left(\sum_{j=1}^{\infty} M_j^{(m)}\right)+R_m+\cdots.
\end{aligned}$$

Since $\|R_n\| \to 0$ as $n\to\infty$ it follows that

$$A=\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} M_j^{(n)}\right).$$

Relabeling the double sequence $\{M_j^{(n)}\}$ as a simple sequence $\{K_j\}$ completes the proof.

(2) A is non-invertible. The proof easily follows from the invertible case. If K has a finite rank, this is known by [1]. If K has an infinite rank, split is as $K=C_1\oplus\cdots C_n\oplus\cdots$ with respect to a decomposition of infinite-dimensional K-invariant subspaces $\mathcal{H}=\mathcal{H}_1\oplus\cdots\mathcal{H}_n\oplus\cdots$, with all $C_n\neq 0$. Split also $A=A_1\oplus\cdots A_n\oplus\cdots$ with respect to a decomposition of A-invariant subspaces $\mathcal{H}=\mathcal{H}'_1\oplus\cdots\mathcal{H}'_n\oplus\cdots$ such that, regarded as an operator on $\mathcal{H}'_n$, A_n is invertible (recall A is nonsingular) and $\|A_n\|_e>1$ for all n. By using a unitary $V:\mathcal{H}\to\mathcal{H}$ such that $V(\mathcal{H}'_n)=\mathcal{H}_n$ we have for each n a pair of operators (VA_nV^*, C_n) on $\mathcal{H}_n$ which satisfies the assumption of the invertible case. We thus obtain a series for each pair (VA_nV^*, C_n) summing up to (VA_nV^*) with general term unitary equivalent to C_n and hence obtain via direct sum a series decomposing VAV^* in the unitary orbit of K. This proves case (2).

II. Now assume $\|A\|_e=1$. If $\|K\|<1$, some scalar multiples of A, K meet the above cases, hence the theorem holds. If $\|K\|=1$, then K does not attain its norm and we can split $K=C_1\oplus\cdots C_n\oplus\cdots$, with respect to a decomposition of infinite-dimensional K-invariant subspaces, where $0<\|C_n\|<1$ for all n. Splitting also $A=A_1\oplus\cdots A_n\oplus\cdots$, with respect to a decomposition of A-invariant subspaces, where $\|A_n\|_e=1$ for all n, and arguing as in case (2) completes the proof. □

3. Some special cases

Corollary 3.1. *Let A, K be two nonzero positive operators.*

1. *A is not compact if and only if there exist $\lambda \in (0, 1]$ and a series whose terms are $\sim$-equivalent to λA and summing up to A.*
2. *If A, K have purely continuous spectra, $0 \in \mathrm{Sp}(K)$ and $\|A\| \geqslant \|K\|$, then there exists a series whose terms are in the unitary orbit of K and summing up to A.*
3. *If A, K have numerical ranges $(0, \alpha)$ and $(0, \kappa)$ respectively, then, $\alpha \geqslant \kappa$ if and only if there exists a series whose terms are in the unitary orbit of K and summing up to A.*
4. *If A does not attain its norm, there exists a series whose terms are $\sim$-equivalent to A and summing up to A. If further $0 \in \mathrm{Sp}_e(A)$, this holds for the $\simeq$-equivalence.*
5. *If K is a strict contraction and $0 \in \mathrm{Sp}_e(K)$, then there exists a sequence of projections $\{P_j\}_{j=1}^{\infty}$ on a larger space $\mathcal{H}' \supset \mathcal{H}$, summing up to the identity on $\mathcal{H}'$, whose compressions $(P_j)_{\mathcal{H}}$ are in the unitary orbit of K for all $j \in \mathbb{N}^*$.*

Statements 1–4 may be derived from Theorem 1.1 (the first statement also follows from [1]). The fifth statement follows from Naimark's Dilation Theorem and Theorem 1.1 with A as the identity on $\mathcal{H}$. Theorem 1.1 with A as the identity can be stated as: *Given any positive strict contraction K with 0 in its essential spectrum, there exists a POVM (positive operator valued measure; i.e., a sequence of positive operators summing up to the identity) whose terms are all unitarily equivalent to K.* Naimark's theorem asserts that such a POVM can be lifted to a total sequence of mutually orthogonal projections on a larger space.

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