C. R. Acad. Sci. Paris, Ser. I 352 (2014) 541–545

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Combinatorics

The ($\leqslant 6$)-half-reconstructibility of digraphs

La ($\leqslant 6$)-demi-reconstructibilité des graphes orientés

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A R T I C L E   I N F O

Article history:
Received 29 January 2011
Accepted after revision 10 March 2014
Available online 19 June 2014

Presented by the Editorial Board

A B S T R A C T

Let $G = (V, A)$ be a digraph. With every subset X of V, we associate the subdigraph $G[X] = (X, A \cap (X \times X))$ of G induced by X. Given a positive integer k, a digraph G is ($\leqslant k$)-half-reconstructible if it is determined up to duality by its subdigraphs of cardinality $\leqslant k$. In 2003, J. Dammak characterized the ($\leqslant k$)-half-reconstructible finite digraphs, for $k \in \{7, 8, 9, 10, 11\}$. N. El Amri extended J. Dammak's characterization to infinite digraphs. In this paper, we characterize the ($\leqslant 6$)-half-reconstructible digraphs.

R É S U M É

Soit $G = (V, A)$ un graphe orienté. À toute partie X de V, on associe le sous-graphe orienté $G[X] = (X, A \cap (X \times X))$ de G induit par X. Étant donné un entier naturel non nul k, un graphe orienté G est ($\leqslant k$)-demi-reconstructible s'il est déterminé à la dualité près par ses sous-graphes de cardinalité $\leqslant k$. En 2003, J. Dammak a caractérisé les graphes orientés finis qui sont ($\leqslant k$)-demi-reconstructibles, pour $k \in \{7, 8, 9, 10, 11\}$. Ensuite, N. El Amri a étendu la caractérisation de J. Dammak pour les graphes orientés infinis. Dans cette note, nous caractérisons les graphes orientés ($\leqslant 6$)-demi-reconstructibles.

Version française abrégée

Un *graphe orienté* est un couple $G = (V, A)$ dans lequel V est un ensemble fini ou infini, appelé ensemble des sommets de G, et A est un ensemble de couples de sommets distincts de G, appelé ensemble des arcs de G. *Le cardinal de* G est celui de V ; il est noté par $|V|$. Pour tous $x \neq y \in V$, la notation $x \longrightarrow_G y$ ou $y \longleftarrow_G x$ signifie $(x, y) \in A$ et $(y, x) \notin A$; on dit dans ce cas que la paire $\{x, y\}$ est une *arête orientée*. Sinon, $\{x, y\}$ est dite une *arête neutre* : elle est *vide* si $(x, y) \notin A$ et $(y, x) \notin A$ – dans ce cas, on note $x \ldots_G y$ – ou *pleine* si $(x, y) \in A$ et $(y, x) \in A$ – dans ce cas, on note $x \longleftrightarrow_G y$. À chaque partie X de V est associé le *sous-graphe orienté* $G[X] = (X, A \cap (X \times X))$ de G induit par X. Le *dual* de G est le graphe orienté $G^* = (V, A^*)$ où $A^* = \{(x, y){:}\ (y, x) \in A\}$. Un *tournoi* $T = (V, A)$ est un graphe orienté vérifant, pour tous $x, y \in V$, $(x, y) \in A$ si et seulement si $(y, x) \notin A$. Un *isomorphisme* de $G = (V, A)$ sur un graphe orienté $G' = (V', A')$ est une bijection f de V sur V' telle que, pour tous $x, y \in V$, $(x, y) \in A$ si et seulement si $(f(x), f(y)) \in A'$. Lorsqu'un tel isomorphisme existe, on

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http://dx.doi.org/10.1016/j.crma.2014.03.009

dit que G et G' sont *isomorphes*; on note $G \simeq G'$. Un graphe orienté est *autodual* s'il est isomorphe à son dual. On dit que G est *hémimorphe* à G' si G' est isomorphe à G ou à G^*.

Étant donné un graphe orienté $G = (V, A)$, on définit une relation d'équivalence $\equiv$ sur V comme suit : pour tout $x \in V$, $x \equiv x$ et pour $x \neq y \in V$, $x \equiv y$ s'il existe une suite de sommets $x_0 = x, x_1, \dots, x_n = y$ vérifiant $x_i \longrightarrow_G x_{i+1}$ ou $x_{i+1} \longrightarrow_G x_i$, pour tout $i \in \{0, \dots, n-1\}$. Lorsque V est non vide, les classes de la relation $\equiv$ sont appelées les *composantes connexes orientées* de G. Une partie I de V est un *intervalle* de G si, pour tout $x \in V - I$, ou bien $x \longrightarrow_G I$, ou $x \longleftarrow_G I$, ou $x \longleftrightarrow_G I$, ou $x \cdots_G I$. Par exemple, $\emptyset$, V et $\{x\}$ (où $x \in V$) sont des intervalles de G appelés les *intervalles triviaux*. Un graphe orienté est *indécomposable* si tous ses intervalles sont triviaux et il est *décomposable* dans le cas contraire. Lorsque G admet un sous-graphe orienté fini non autodual, on note $\mathcal{C}_{\text{dual}}(G)$ le plus petit cardinal des sous-graphes non autoduaux de G. Un *drapeau* est un graphe orienté hémimorphe à $(\{0,1,2\}, \{(0,1), (1,2), (2,1)\})$. Un *pic* est un graphe orienté hémimorphe à $(\{0,1,2\}, \{(0,1), (0,2), (1,2), (2,1)\})$ ou à $(\{0,1,2\}, \{(0,1), (0,2)\})$. Un *diamant* est un tournoi hémimorphe à $(\{0,1,2,3\}, \{(0,1), (0,2), (0,3), (1,2), (2,3), (3,1)\})$. Une *préchaîne* est un graphe orienté G qui n'admet aucun sous-graphe orienté isomorphe à un pic ou à un diamant et tel que ses éventuelles arêtes neutres sont deux à deux disjointes. Si G est un tournoi, G est simplement un *tournoi sans diamant*.

Considérons deux graphes orientés G et G' de même ensemble de sommets V à v éléments et un entier naturel k; G et G' sont $(\leqslant k)$*-hypomorphes* (resp. $(\leqslant k)$*-hémimorphes*) si, pour toute partie X de V telle que $|X| \leqslant k$, $G[X]$ et $G'[X]$ sont isomorphes (resp. hémimorphes). Le graphe orienté G est $(\leqslant k)$*-reconstructible* (resp. $(\leqslant k)$*-demi-reconstructible*) si tout graphe orienté $(\leqslant k)$-hypomorphe (resp. $(\leqslant k)$-hémimorphe) à G lui est isomorphe (resp. hémimorphe). La notion de la demi-reconstruction a été introduite en 1993 par J.G. Hagendorf. Le problème de la $(\leqslant k)$-demi-reconstruction est résolu par J.G. Hagendorf et G. Lopez [16]. En fait, ils ont prouvé que les graphes orientés finis sont $(\leqslant 12)$-demi-reconstructibles. Y. Boudabbous et G. Lopez [9] ont montré que si deux tournois finis T et T' sont $(\leqslant 7)$-hémimorphes, alors T et T' sont hémimorphes. Pour les graphes orientés connexes finis, J. Dammak a prouvé qu'ils sont $(\leqslant 7)$-demi-reconstructibles [11], et qu'ils sont (-5)-demi-reconstructibles [13]. Il a également montré que, pour $k \in \{3,4,5,6\}$, un graphe orienté fini est $(\leqslant k+6)$-demi-reconstructible, à condition qu'un de ses sous-graphes orientés à k éléments ne soit pas autodual [11]. M. Pouzet [1,2] a posé le problème de la $\{-k\}$-reconstruction. P. Ille [17] (resp. G. Lopez et C. Rauzy [20,21]) ont prouvé que les graphes orientés finis d'au moins 11 (resp. 10) sommets sont $\{-5\}$-reconstructibles (resp. $\{-4\}$-reconstructibles). En 1998, Y. Boudabbous et J. Dammak [7] ont introduit la $\{-k\}$-demi-reconstruction et ont prouvé que, pour $k \in \{4,5\}$, les tournois finis d'au moins $k+12$ sommets sont $\{-k\}$-demi-reconstructibles. En 2012, Y. Boudabbous et C. Delhommé ont étudié l'autodualité des graphes orientés (finis et infinis) [8]. Dans leur étude, ils ont introduit la notion des préchaînes. En 2003, J. Dammak [12] a caractérisé les graphes orientés finis qui sont $(\leqslant k)$-demi-reconstructibles, pour $k \in \{7,8,9,10,11\}$. Dans le cas des tournois, Y. Boudabbous, A. Boussairi, A. Chaïchaâ et N. El Amri [6] ont caractérisé les tournois finis qui sont $(\leqslant k)$-demi-reconstructibles, pour $k \in \{3,4,5,6\}$. Ensuite, N. El Amri [14] a étendu la caracérisation de J. Dammak aux graphes orientés infinis.

Soit I un intervalle propre de $G = (V, A)$. Le *graphe orienté contracté* de G en I est le graphe orienté $G_I = ((V - I) \cup \{I\}, A_I)$, où A_I est défini comme suit : $(x, y) \in A_I$ si $[(x, y) \in A \cap (V - I) \times (V - I)]$ ou $[x = I, y \notin I$ et $\exists z \in I\colon (z, y) \in A]$ ou $[x \notin I, y = I$ et $\exists z \in I\colon (x, z) \in A]$. Plus précisément, G_I est le graphe orienté obtenu à partir de G, en considérant I comme un sommet. Notre principal résultat est :

Théorème 1. *Soit G un graphe orienté $(\leqslant 7)$-demi-reconstructible. Le graphe G est non $(\leqslant 6)$-demi-reconstructible si et seulement si l'une des conditions suivantes est satisfaite :*

- L_1. *G possède au moins deux intervalles I_1 et I_2 qui sont des tournois sans diamant non autoduaux et ne sont pas des composantes connexes orientées.*
- L_2. *G a exactement un intervalle I_0 non autodual qui est un tournoi sans diamant et qui n'est pas une composante connexe orientée. De plus, il n'existe aucun isomorphisme f de G_{I_0} sur $G^*_{I_0}$ vérifiant $f(I_0) = I_0$.*
- L_3. *$\mathcal{C}_{\text{dual}}(G) = 3$, G possède au moins deux composantes connexes orientées non autoduales D_1, D_2, qui sont des intervalles de G et sont ou bien disjointes de tout drapeau, avec $\mathcal{C}_{\text{dual}}(G[D_i]) = 4$ pour $i \in \{1,2\}$, ou bien des préchaînes non tournois et pour chaque $i \in \{1,2\}$, D_i contient un sommet a_i d'un drapeau $\{a_i, b_i, c_i\}$ avec $b_i, c_i \notin D_i$ et $\{b_i, c_i\}$ est l'arête orientée.*
- L_4. *$\mathcal{C}_{\text{dual}}(G) = 3$, G a exactement une composante connexe orientée non autoduale D_0 qui est un intervalle, et est ou bien disjointe de tout drapeau avec $\mathcal{C}_{\text{dual}}(G[D_0]) = 4$, ou bien une préchaîne non tournoi et D_0 contient un sommet a_0 d'un drapeau $\{a_0, b_0, c_0\}$ avec $b_0, c_0 \notin D_0$ et $\{b_0, c_0\}$ est une arête orientée. En plus, il n'existe aucun isomorphisme f de G_{D_0} sur $G^*_{D_0}$ tel que $f(D_0) = D_0$.*

Si la condition L_1 du Théorème 3 est satisfaite, nécessairement I_1 et I_2 sont disjoints et non triviaux. Aussi, dans la condition L_3, la paire orientée $\{b_i, c_i\}$ est disjointe de D_j, où $\{j\} = \{1,2\} - \{i\}$.

1. Introduction

A *directed graph* or simply *digraph* G consists of a set $V(G)$ of vertices together with a prescribed collection $A(G)$ of ordered pairs of distinct vertices, called the set of the *arcs* of G. Such a digraph is denoted by $(V(G), A(G))$ or simply by (V, A). *The cardinality of* G is that of V. We denote this cardinality by $|V(G)|$ as well as $|G|$. Given a digraph $G = (V, A)$, the *dual* of G is the digraph $G^* = (V, A^*)$ defined by: for $x \neq y \in V$, $(x, y) \in A^*$ if $(y, x) \in A$. With each subset X of V is associated the *subdigraph* $(X, A \cap (X \times X))$ of G *induced* by X, denoted by $G[X]$. The subdigraph $G[V - X]$ is also denoted by $G - X$. For $x \neq y \in V$, $x \longrightarrow_G y$ or $y \longleftarrow_G x$ means $(x, y) \in A$ and $(y, x) \notin A$, $x \longleftrightarrow_G y$ means $(x, y) \in A$ and $(y, x) \in A$, $x \ldots_G y$ means $(x, y) \notin A$ and $(y, x) \notin A$. For $x \in V$ and $Y \subseteq V$, $x \longrightarrow_G Y$ signifies that for every $y \in Y$, $x \longrightarrow_G y$. For X, $Y \subseteq V$, $X \longrightarrow_G Y$ (or simply $X \longrightarrow Y$ or $X < Y$ if there is no danger of confusion) signifies that for every $x \in X$, $x \longrightarrow_G Y$. For $x \in V$ and for X, $Y \subseteq V$, $x \longleftarrow_G Y$, $x \longleftrightarrow_G Y$, $x \ldots_G Y$, $X \longleftrightarrow_G Y$ and $X \ldots_G Y$ are defined in the same way. Given a digraph $G = (V, A)$, two distinct vertices x and y of G form a *directed pair* if either $x \longrightarrow_G y$ or $x \longleftarrow_G y$. Otherwise, $\{x, y\}$ is a *neutral pair*; it is *full* if $x \longleftrightarrow_G y$, and *void* when $x \ldots_G y$.

A digraph $T = (V, A)$ is a *tournament* if all its pairs of vertices are directed. A *transitive tournament* is a tournament T such that for $x, y, z \in V(T)$, if $x \longrightarrow_T y$ and $y \longrightarrow_T z$ then $x \longrightarrow_T z$. This is simply a *chain* (that is a set equipped with a linear order in which the loops have been deleted). Hence, we will consider the chain ω of non-negative integers as a transitive tournament as well as the chain $\omega^* + \omega$ of integers. A *flag* is a digraph hemimorphic to $(\{0, 1, 2\}, \{(0, 1), (1, 2), (2, 1)\})$. A *peak* is a digraph hemimorphic to $(\{0, 1, 2\}, \{(0, 1), (0, 2), (1, 2), (2, 1)\})$ or to $(\{0, 1, 2\}, \{(0, 1), (0, 2)\})$. A *diamond* is a digraph hemimorphic to $(\{0, 1, 2, 3\}, \{(0, 1), (0, 2), (0, 3), (1, 2), (2, 3), (3, 1)\})$.

A *prechain* is a digraph that embeds neither peak nor diamond nor adjacent neutral pairs. Clearly, a chain is a prechain. A *proper prechain* is a prechain that is not a chain. A prechain which is a tournament is a *diamond-free tournament* (that is simply a tournament with no diamond). Call a *finite consecutivity* (resp. an *infinite consecutivity*), each digraph on at least three vertices isomorphic to one of the digraphs gotten from a finite chain, (resp. a transitive tournament of type ω, ω^* or $\omega^* + \omega$) such that the pairs of non-consecutive vertices become either all full or all void. A consecutivity obtained from ω or ω^* is called also *one-end infinite consecutivity*. A *cycle* is any digraph isomorphic to one of the digraphs obtained from a finite consecutivity on $n \geqslant 3$ vertices by replacing the neutral pair $\{0, n - 1\}$ by $(n - 1) \longrightarrow 0$, where 0 and $n - 1$ are the initial and the final extremity respectively. Clearly every 3-*cycle* is isomorphic to the tournament $(\{0, 1, 2\}, \{(0, 1), (1, 2), (2, 0)\})$.

Given a digraph $G = (V, A)$, we define an equivalence relation $\equiv$ on V as follows: for all $x \in V$, $x \equiv x$ and for $x \neq y \in V$, $x \equiv y$ if there is a sequence $x_0 = x, \ldots, x_n = y$ of vertices of G fulfilling that: either $x_i \longrightarrow_G x_{i+1}$ or $x_i \longleftarrow_G x_{i+1}$, for all $i \in \{0, \ldots, n - 1\}$. If $V \neq \emptyset$, the $\equiv$'s classes are called *arc-connected components* of G. A digraph is said to be *arc-connected* if it has at most one arc-connected component.

Given a digraph $G = (V, A)$, a subset I of V is an *interval* of G if for every $x \in V - I$ either $x \longrightarrow_G I$ or $x \longleftarrow_G I$ or $x \longleftrightarrow_G I$ or $x \cdots_G I$. For instance, $\emptyset$, V and $\{x\}$ (where $x \in V$) are intervals of G, called *trivial intervals*. A digraph is *indecomposable* if all its intervals are trivial, otherwise it is *decomposable*.

Given two digraphs $G = (V, A)$ and $G' = (V', A')$, a bijection f from V onto V' is an *isomorphism* from G onto G' provided that for any $x, y \in V$, $(x, y) \in A$ if and only if $(f(x), f(y)) \in A'$. The digraphs G and G' are then *isomorphic*, which is denoted by $G \simeq G'$, if there exists an isomorphism from G onto G'. If G and G' are not isomorphic, we write $G \not\simeq G'$. For instance, G and G' are *hemimorphic*, if G' is isomorphic to G or to G^*. A digraph G is said to be *self-dual* if G is isomorphic to G^*. A digraph H *embeds into* a digraph G if H is isomorphic to a subdigraph of G.

Consider two digraphs G' and G on the same vertex set V with v elements and a positive integer k. The digraphs G' and G are $\{k\}$-*hypomorphic* (resp. $\{k\}$-*hemimorphic* [3]) whenever for every subset X of V with $|X| = k$, the subdigraphs $G'[X]$ and $G[X]$ are isomorphic (resp. hemimorphic). G' and G are $\{-k\}$-hypomorphic whenever either $k > v$ or $k \leqslant v$ and G' and G are $\{v - k\}$-hypomorphic. Notice that G and G' are trivially $\{0\}$-hypomorphic, however G and G' are $\{-0\}$-hypomorphic if and only if they are isomorphic.

A digraph G is $\{-k\}$-*self-dual* if it is $\{-k\}$-*hypomorphic* to G^*. Let F be a set of integers. The digraphs G and G' are F-*hypomorphic* (resp. F-*hemimorphic*), if for every $k \in F$, the digraphs G and G' are $\{k\}$-hypomorphic (resp. $\{k\}$-hemimorphic). The digraph G is F-reconstructible (resp. F-half-reconstructible) provided that every digraph F-hypomorphic (resp. F-hemimorphic) to G is isomorphic (resp. hemimorphic) to G. The digraphs G and G' are $(\leqslant k)$-*hypomorphic* (resp. $(\leqslant k)$-hemimorphic) if they are $\{1, \ldots, k\}$-hypomorphic (resp. $\{1, \ldots, k\}$-hemimorphic). The digraphs G and G' are *hereditarily isomorphic* (resp. *hereditarily hemimorphic*) if for all $X \subseteq V$, $G[X]$ and $G'[X]$ are isomorphic (resp. hemimorphic).

Let $G = (V, A)$ and $G' = (V, A')$ be two $(\leqslant 2)$-hemimorphic digraphs. Denote by $\mathfrak{D}_{G,G'}$ the binary relation on V such that: for $x \in V$, $x \mathfrak{D}_{G,G'} x$; and for $x \neq y \in V$, $x \mathfrak{D}_{G,G'} y$ if there exists a sequence $x_0 = x, \ldots, x_n = y$ of elements of V satisfying $(x_i, x_{i+1}) \in A$ if and only if $(x_i, x_{i+1}) \notin A'$, for all i, $0 \leqslant i \leqslant n - 1$. The relation $\mathfrak{D}_{G,G'}$ is an equivalence relation called *the difference relation*, its classes are called *difference classes*. Let denote by $D_{G,G'}$ the set of difference classes.

The $(\leqslant k)$-reconstruction was introduced by R. Fraïssé in 1970 [15]. In 1972, G. Lopez [18,19] introduced the difference relation and showed that:

Theorem 1. *(See [18,19].) The finite digraphs are* $(\leqslant 6)$*-reconstructible (i.e.: if G and G′ are* $(\leqslant 6)$*-hypomorphic, then G and G′ are isomorphic).*

In 2002, the $(\leqslant 5)$-reconstructibility of finite digraphs was studied by Y. Boudabbous [4]. For $k \in \{3, 4\}$, the $(\leqslant k)$-reconstructibility of finite digraphs was studied by Y. Boudabbous and G. Lopez [10] in 2005. In 1993, J.G. Hagendorf raised the $(\leqslant k)$-half-reconstruction and solved it with G. Lopez [16]: they proved that, if two digraphs G and G' are $(\leqslant 12)$-hemimorphic, then either G' and G or G' and G^* are $(\leqslant 6)$-hypomorphic. From that, they obtained in particular: *The finite digraphs are* $(\leqslant 12)$*-half-reconstructible*. Y. Boudabbous and G. Lopez [9] showed that if two finite tournaments T and T' are $(\leqslant 7)$-hemimorphic, then T and T' are hemimorphic. Concerning the finite arc-connected digraphs, J. Dammak proved that they are $(\leqslant 7)$-half-reconstructible [11], and that they are (-5)-half-reconstructible [13]. He also shown that finite digraphs embedding a non-self-dual subdigraph of cardinality k, are $(\leqslant k+6)$-half-reconstructible, for $k \in \{3,4,5,6\}$ [11]. M. Pouzet [1,2] introduced the $\{-k\}$-reconstructibility. P. Ille [17] (resp. G. Lopez and C. Rauzy [20,21]) proved that the finite digraphs on at least 11 (resp. 10) vertices are $\{-5\}$-reconstructible (resp. $\{-4\}$-reconstructible). Y. Boudabbous [5] improved that: for $k \in \{4,5\}$, two $\{-k\}$-hypomorphic finite tournaments, on at least $k+6$ vertices, are hereditarily isomorphic. In 1998, Y. Boudabbous and J. Dammak [7] introduced the $\{-k\}$-half-reconstruction and proved that: for $k \in \{4,5\}$, the finite tournaments with at least $k+12$ vertices are $\{-k\}$-half-reconstructible. In 2012, Y. Boudabbous and C. Delhommé [8] studied self-duality and introduced the notion of prechain. In 2003, J. Dammak [12] characterized finite digraphs which are $(\leqslant k)$-half-reconstructible, for $k \in \{7,8,9,10,11\}$. After that, N. El Amri [14], extended J. Dammak's characterization to infinite digraphs. In the case of tournaments, Y. Boudabbous, A. Boussairi, A. Chaïchaâ and N. El Amri [6] characterized finite tournaments which are $(\leqslant k)$-half-reconstructible, for $k \in \{3,4,5,6\}$.

Let $G=(V,A)$ be a digraph and I be a proper interval of G. We call *contracted digraph* of G into I, the digraph $G_I = ((V-I)\cup\{I\}, A_I)$, where A_I is defined as follows: $(x,y)\in A_I$ if $[(x,y)\in A\cap(V-I)\times(V-I)]$ or $[x=I, y\notin I$ and $\exists z\in I\colon (z,y)\in A]$ or $[x\notin I, y=I$ and $\exists z\in I\colon (x,z)\in A]$. More precisely, G_I is the digraph obtained from G by considering I as a vertex.

If G satisfies one of the following conditions, we say that G satisfies the condition $\mathcal{C}_\infty$

H_1. G has at least an infinite chain interval.
H_2. G has at least two one-end infinite consecutivity intervals.
H_3. G has exactly a unique one-end infinite consecutivity interval I and there is no isomorphism f from G_I onto G_I^* such that $f(I)=I$.

N. El Amri [14] proved that a digraph is non-$(\leqslant 12)$-half-reconstructible if and only if it verifies $\mathcal{C}_\infty$.

Given a digraph G with a non-self-dual finite subdigraph, $\mathcal{C}_{\text{dual}}(G)$ denotes the smallest cardinality of the non-self-dual finite subdigraphs of G. From Theorem 1, $3\leqslant \mathcal{C}_{\text{dual}}(G)\leqslant 6$. In the case where G has no non-self-dual finite subdigraph, we set $\mathcal{C}_{\text{dual}}(G)=\infty$.

Clearly, all non-$(\leqslant 12)$-half-reconstructible digraphs are not $(\leqslant k)$-half-reconstructible, for $k\in\{6,7,8,9,10,11\}$.

Theorem 2. *(See [14].) Let G be a $(\leqslant 12)$-half-reconstructible digraph. The digraph G is non-$(\leqslant 7)$-half-reconstructible if and only if one of the following conditions holds:*

K_1. *$\mathcal{C}_{\text{dual}}(G)=3$ and G admits at least two non-self-dual arc-connected components which are intervals of type diamond-free tournament or non-tournament prechain disjoint from any flag.*
K_2. *$\mathcal{C}_{\text{dual}}(G)=3$ and G has exactly one non-self-dual arc-connected component D_0, which is a diamond-free tournament or a non-tournament prechain disjoint from any flag, and there is no isomorphism f from G_{D_0} onto $G^*_{D_0}$ such that $f(D_0)=D_0$.*
K_3. *$\mathcal{C}_{\text{dual}}(G)=4$ and G admits at least two non-self-dual arc-connected components which are intervals.*
K_4. *$\mathcal{C}_{\text{dual}}(G)=5$ and G admits at least two non-self-dual arc-connected components which are prechain intervals.*
K_5. *$\mathcal{C}_{\text{dual}}(G)=6$ and G admits at least two non-self-dual arc-connected components which are diamond-free tournament intervals.*

As each non-$(\leqslant 7)$-half-reconstructible digraph is not $(\leqslant 6)$-half-reconstructible, we obtain our main result:

Theorem 3. *Let G be a $(\leqslant 7)$-half-reconstructible digraph. The digraph G is non-$(\leqslant 6)$-half-reconstructible if and only if one of the following conditions holds:*

L_1. *G has at least two intervals I_1 and I_2 which are non-self-dual diamond-free tournaments and are not arc-connected components.*
L_2. *G has exactly one non-self-dual interval I_0 which is a diamond-free tournament that is not an arc-connected component. Furthermore, there is no isomorphism f from G_{I_0} onto $G^*_{I_0}$ such that $f(I_0)=I_0$.*
L_3. *$\mathcal{C}_{\text{dual}}(G)=3$ and G has at least two non-self-dual arc-connected components D_1, D_2 which are intervals and either disjoint from any flag such that $\mathcal{C}_{\text{dual}}(G[D_i])=4$, for $i\in\{1,2\}$, or these intervals are non-tournament prechain and each D_i contains a vertex a_i of a flag $\{a_i,b_i,c_i\}$ with $b_i,c_i\notin D_i$ and $\{b_i,c_i\}$ is a directed pair.*
L_4. *$\mathcal{C}_{\text{dual}}(G)=3$ and G has exactly one non-self-dual arc-connected component D_0 which is an interval being either disjoint from any flag such that $\mathcal{C}_{\text{dual}}(G[D_0])=4$, or is a non-tournament prechain containing a vertex a_0 of a flag $\{a_0,b_0,c_0\}$ with $b_0,c_0\notin D_0$ and $\{b_0,c_0\}$ is a directed pair. Furthermore, there is no isomorphism f from G_{D_0} onto $G^*_{D_0}$ such that $f(D_0)=D_0$.*

If the condition L_1 of Theorem 3 is satisfied, necessarily I_1 and I_2 are disjoint and nontrivial.
Also, in condition L_3, the directed pair $\{b_i,c_i\}$ is disjoint from D_j where $\{j\}=\{1,2\}-\{i\}$.

2. Proof of Theorem 3

The proof of Theorem 3 uses essentially the following results.

Proposition 1. *Let G be a $(\leqslant 7)$-half-reconstructible arc-connected digraph. The digraph G is non-$(\leqslant 6)$-half-reconstructible if and only if G verifies condition L_1 or L_2 of Theorem 3.*

Lemma 1. *Let $G=(V,A)$ and $G'=(V,A')$ be two $(\leqslant 6)$-hemimorphic digraphs and D be an arc-connected component of G. Let $I_0 \subset V$, such that $|I_0| = \mathcal{C}_{\text{dual}}(G)$, $G[I_0]$ is not self-dual and $G'[I_0] \simeq G[I_0]$.*

1. *If $C \in D_{G,G'}$, then C is an interval of G and G'.*
2. *If D is not an interval of G, then $\mathfrak{D}_{G[D],G'[D]}$ has at least two equivalence classes.*
3. *If $\mathcal{C}_{\text{dual}}(G[D]) = 3$, then $\mathfrak{D}_{G[D],G'[D]}$ has at least two equivalence classes.*

Lemma 2. *Let $G=(V,A)$ and $G'=(V,A')$ be two $(\leqslant 6)$-hemimorphic digraphs such that G does not satisfy the condition $\mathcal{C}_\infty$ and D be an arc-connected component of G. Let $I_0 \subset V$, such that $|I_0| = \mathcal{C}_{\text{dual}}(G)$, $G[I_0]$ is not self-dual and $G'[I_0] \simeq G[I_0]$.*

1. *Let $C \in D_{G[D],G'[D]}$, such that $G[C]$ is neither a one-end infinite consecutivity nor a non-self-dual diamond-free tournament. If C is different from its arc-connected component, then C is an interval of G and G', and $G'[C] \simeq G[C]$.*
2. *If $G[D]$ has no interval which is a one-end infinite consecutivity or a non-self-dual diamond-free tournament and if $\mathfrak{D}_{G[D],G'[D]}$ has at least two equivalence classes, then for every $C \in D_{G[D],G'[D]}$, C is an interval of G and G', and $G'[C] \simeq G[C]$. So, $G'[D] \simeq G[D]$.*

Conversely, assuming that $G=(V,A)$ does not verify L_1, L_2, L_3, and L_4, we will prove that G is $(\leqslant 6)$-half-reconstructible. As G is $(\leqslant 7)$-half-reconstructible, G is $(\leqslant 12)$-half-reconstructible. So, in the sequel, the digraphs considered do not satisfy any of the conditions C_∞, K_1, K_2, K_3, K_4, K_5, L_1, L_2, L_3, and L_4.

Lemma 3. *Let $G'=(V,A')$ be a digraph $(\leqslant 6)$-hemimorphic to G and I_0 be a subset of V, such that $G[I_0]$ is a peak or a flag and $G'[I_0] \simeq G[I_0]$. Assume that G has an interval M_0 which is either a one-end infinite consecutivity or a non-self-dual diamond-free tournament. Let D be an arc-connected component disjoint from M_0. Then,*

1. *There exists an isomorphism f from G_{M_0} onto $G^*_{M_0}$ such that $f(M_0) = M_0$.*
2. *D has not an interval which is either a one-end infinite consecutivity or a non-self-dual diamond-free tournament.*
3. *$\mathfrak{D}_{G[D],G'[D]}$ has at least two equivalence classes or D is self-dual.*
4. *G and G' are hemimorphic.*

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