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Number theory

The elementary symmetric functions of a reciprocal polynomial sequence

Les fonctions symétriques élémentaires des suites d'inverses de valeurs de polynômes

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ABSTRACT

Erdös and Niven proved in 1946 that for any positive integers m and d, there are at most finitely many integers n for which at least one of the elementary symmetric functions of $1/m, 1/(m+d), \dots, 1/(m+(n-1)d)$ are integers. Recently, Wang and Hong refined this result by showing that if $n \geqslant 4$, then none of the elementary symmetric functions of $1/m, 1/(m+d), \dots, 1/(m+(n-1)d)$ is an integer for any positive integers m and d. Let f be a polynomial of degree at least 2 and of nonnegative integer coefficients. In this paper, we show that none of the elementary symmetric functions of $1/f(1), 1/f(2), \dots, 1/f(n)$ is an integer except for $f(x) = x^m$ with $m \geqslant 2$ being an integer and $n = 1$.

RÉSUMÉ

Erdös et Niven ont démontré en 1946 que, pour tous entiers positifs m et d, il n'y a qu'un nombre fini d'entiers positifs n pour lesquels au moins une des fonctions symétriques élémentaires des nombres $1/m, 1/(m+d), \dots, 1/(m+(n-1)d)$ est entière. Récemment, Wang et Hong ont raffiné ce résultat en montrant que, si $n \geqslant 4$, alors aucune des fonctions symétriques élémentaires des nombres $1/m, 1/(m+d), \dots, 1/(m+(n-1)d)$ n'est entière, pour tous entiers positifs m et d. Soit f un polynôme de degré au moins 2 et à coefficients entiers positifs ou nuls. Nous établissons dans cette Note qu'aucune des fonctions symétriques élémentaires des nombres $1/f(1), 1/f(2), \dots, 1/f(n)$ n'est entière, sauf si $f(x) = x^m$ avec $m \geqslant 2$ entier et $n = 1$.

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1. Introduction

Let n be a positive integer and $f(x)$ be a polynomial of integer coefficients such that $f(m) \neq 0$ for any integer $m \geqslant 1$. For any integer k with $1 \leqslant k \leqslant n$, we denote by $\sigma_{k,f}(n)$ the k-th elementary symmetric function of $1/f(1), 1/f(2), \ldots, 1/f(n)$. That is,

$$\sigma_{k,f}(n) := \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} \prod_{j=1}^{k} \frac{1}{f(i_j)}.$$

A well-known result says that if $n \geqslant 2$ and $f(x) = x$, then the harmonic sum $\sigma_{1,f}(n)$ cannot be an integer. More generally, if $n \geqslant 2$ and $f(x) = ax + b$ with a and b being positive integers, then the sum $\sigma_{1,f}(n)$ is not an integer. In 1946, Erdös and Niven [2] extended this result by showing that if $f(x) = ax + b$ with a and b being positive integers, then there are at most finitely many integers n for which at least one element in the set $S(f,n) := \{\sigma_{1,f}(n), \sigma_{2,f}(n), \ldots, \sigma_{n,f}(n)\}$ is an integer. In 2012, Chen and Tang [1] proved that each element of $S(f,n)$ is not an integer if $f(x) = x$ and $n \geqslant 4$. Wang and Hong [4] showed that none of the elements in $S(f,n)$ is an integer if $f(x) = 2x - 1$ and $n \geqslant 2$. Recently, Wang and Hong [5] refined the theorem of Erdös and Niven [2] by showing that if $f(x) = ax + b$ with a and b being positive integers and $n \geqslant 4$, then all the elements in $S(f,n)$ are not integers. An interesting problem naturally arises: does the similar result hold when $f(x)$ is a polynomial of nonnegative integer coefficients and of degree at least two?

In this paper, our main goal is to answer the above problem. In fact, we determine all the finite progressions $\{f(i)\}_{i=1}^{n}$ with $f(x)$ being of nonnegative coefficients such that one or more elements in $S(f,n)$ are integers. In other words, we have the following result.

Theorem 1.1. *Let f be a polynomial of nonnegative integer coefficients and of degree at least two. Let n and k be integers such that $1 \leqslant k \leqslant n$. Then $\sigma_{k,f}(n)$ is not an integer except for the case $f(x) = x^m$ with $m \geqslant 2$ being an integer and $k = n = 1$, in which case, $\sigma_{k,f}(n)$ is an integer.*

Evidently, Theorem 1.1 answers completely the above problem. In the next section, we will give the proof of Theorem 1.1. A conjecture is proposed in the last section.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. To do so, we first list two known identities about the values of Riemann zeta function at 2 and 4 (see, for example, [3]):

$$\zeta(2) = \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} \quad \text{and} \quad \zeta(4) = \sum_{j=1}^{\infty} \frac{1}{j^4} = \frac{\pi^4}{90}.$$

Then we can easily see that $1 < \zeta(2) < 2$. Notice that $\sigma_{k,f}(n) > 0$ for any integer $n \geqslant 1$.

We can now give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $f(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0$ with $a_m \geqslant 1$ and $m \geqslant 2$ being integers. First we let $k \geqslant 2$. It follows from the hypotheses $a_m \geqslant 1$ and $m \geqslant 2$ that $f(r) \geqslant r^2$ for any positive integer r. Since $\zeta(2) < 2$, we deduce that

$$\begin{aligned}
\sigma_{k+1,f}(n) &= \sum_{1 \leqslant i_1 < \cdots < i_{k+1} \leqslant n} \prod_{j=1}^{k+1} \frac{1}{f(i_j)} \\
&= \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n-1} \left(\prod_{j=1}^{k} \frac{1}{f(i_j)} \right) \left(\sum_{i_{k+1}=i_k+1}^{n} \frac{1}{f(i_{k+1})} \right) \\
&\leqslant \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n-1} \left(\prod_{j=1}^{k} \frac{1}{f(i_j)} \right) \left(\sum_{i_{k+1}=2}^{\infty} \frac{1}{i_{k+1}^2} \right) \\
&= \big(\zeta(2) - 1\big) \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n-1} \prod_{j=1}^{k} \frac{1}{f(i_j)} \\
&= \big(\zeta(2) - 1\big) \sigma_{k,f}(n-1) \\
&< \sigma_{k,f}(n-1) < \sigma_{k,f}(n).
\end{aligned} \tag{2.1}$$

So for any given integer n, $\sigma_{k,f}(n)$ is decreasing as k increases. On the other hand, we have:

$$\begin{aligned}\sigma_{2,f}(n) &= \sum_{1\leqslant i_1<i_2\leqslant n}\frac{1}{f(i_1)f(i_2)}\\ &\leqslant \sum_{1\leqslant i_1<i_2\leqslant n}\frac{1}{i_1^2 i_2^2}\\ &< \sum_{i_2>i_1\geqslant 1}\frac{1}{i_1^2 i_2^2}\\ &= \frac{1}{2}\left(\left(\sum_{j=1}^{\infty}\frac{1}{j^2}\right)^2-\sum_{j=1}^{\infty}\frac{1}{j^4}\right)\\ &= \frac{1}{2}\left(\zeta(2)^2-\zeta(4)\right)=\frac{\pi^4}{120}<1. \end{aligned}\tag{2.2}$$

Thus, by (2.1) and (2.2), we obtain that $0<\sigma_{k,f}(n)<1$ if $2\leqslant k\leqslant n$. This concludes that $\sigma_{k,f}(n)$ is not an integer if $k\geqslant 2$. So Theorem 1.1 is true for the case when $k\geqslant 2$.

In what follows we let $k=1$. First we assume that f contains only one term, namely $f(x)=ax^m$, where $m\geqslant 2$ and $a\geqslant 1$. Clearly, if $a\geqslant 2$, then

$$0<\sigma_{1,f}(n)\leqslant\frac{1}{2}\sum_{j=1}^{n}\frac{1}{j^2}<\frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{j^2}=\frac{\pi^2}{12}<1.$$

If $a=1$, then $f(x)=x^m$. It follows that $\sigma_{1,f}(1)=1$ and

$$1<\sigma_{1,f}(n)\leqslant\sum_{j=1}^{\infty}\frac{1}{j^2}=\frac{\pi^2}{6}<2$$

for any integer $n\geqslant 2$. Hence for any $n\geqslant 2$, $\sigma_{1,f}(n)$ is not an integer if $f(x)=a_mx^m$ with $m\geqslant 2$ and $a_m\geqslant 1$.

Now we suppose that f contains at least two terms. Then one may let $f(x)=a_mx^m+a_{m-1}x^{m-1}+\cdots+a_0$, where $m\geqslant 2, a_m\geqslant 1$ and $\max(a_0,\ldots,a_{m-1})\geqslant 1$. We divide the proof into the following three cases.

Case 1. $m=2$, $a_1=0$, $a_0=a_2=1$. Then $f(x)=x^2+1$. By a simple calculation, we see that $\sigma_{1,f}(12)<1$, $\sigma_{1,f}(13)>1$. So we can conclude that $0<\sigma_{1,f}(n)\leqslant\sigma_{1,f}(12)<1$ if $n\leqslant 12$, and

$$1<\sigma_{1,f}(13)\leqslant\sigma_{1,f}(n)<\sum_{j=1}^{\infty}\frac{1}{j^2+1}<\sum_{j=1}^{\infty}\frac{1}{j^2}=\zeta(2)<2$$

if $n\geqslant 13$. Thus $\sigma_{1,f}(n)$ is not an integer in this case.

Case 2. $m=2$, $a_1=0$ and $\max(a_0,a_2)\geqslant 2$. Then for any positive integer j, one can deduce that $f(j)=a_2j^2+a_0\geqslant j^2+2$. It then follows that

$$0<\sigma_{1,f}(1)\leqslant\frac{1}{3}<1,\qquad 0<\sigma_{1,f}(2)\leqslant\frac{1}{3}+\frac{1}{6}<1$$

and

$$0<\sigma_{1,f}(n)\leqslant\sum_{j=1}^{n}\frac{1}{j^2+2}<\frac{1}{3}+\frac{1}{6}+\sum_{j=3}^{n}\frac{1}{(j-1)j}=\frac{1}{3}+\frac{1}{6}+\frac{1}{2}-\frac{1}{n}<1$$

if $n\geqslant 3$. Namely, $\sigma_{1,f}(n)$ is not an integer in this case.

Case 3. Either $m=2$ and $a_1\geqslant 1$, or $m\geqslant 3$. If $m\geqslant 3$, since $f(x)$ contains at least two terms, it follows that there is an integer l with $0\leqslant l<m$ such that $a_l\geqslant 1$. Hence for any positive integer j, we derive that

$$f(j)\geqslant a_mj^m+a_lj^l\geqslant j^3+1\geqslant j^2+j$$

if $m\geqslant 3$. If $m=2$ and $a_1\geqslant 1$, then for any positive integer j, we have $f(j)=a_2j^2+a_1j+a_0\geqslant j^2+j$. Based on the above discussions, we can deduce that

$$0<\sigma_{1,f}(n)=\sum_{j=1}^{n}\frac{1}{f(j)}\leqslant\sum_{j=1}^{n}\frac{1}{j^2+j}=1-\frac{1}{n+1}<1.$$

So $\sigma_{1,f}(n)$ is not an integer in this case.

This completes the proof of Theorem 1.1 for the case that $k=1$. So Theorem 1.1 is proved. □

3. Remarks

In this section, we raise the following conjecture as the conclusion of this paper.

Conjecture 3.1. *Let $f(x)$ be a polynomial of integer coefficients such that $f(m) \neq 0$ for any positive integer m. Then there is a positive integer N such that for any integer $n \geqslant N$ and for all integers k with $1 \leqslant k \leqslant n$, $\sigma_{k,f}(n)$ is not an integer.*

Clearly, by [2] (or [5]) and Theorem 1.1 we know that Conjecture 3.1 is true if $f(x)$ is of nonnegative integer coefficients. Further, by [5] one can derive that Conjecture 3.1 holds if $f(x) = ax - b$, where a and b are integers such that $a > b > 0$. But it is kept open for the case when either $f(x) = ax - b$, with a and b being integers such that $0 < a < b$, or $f(x)$ is of degree greater than 2 and contains negative coefficients, but its leading coefficient is positive.

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