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Topology

On the vanishing of the Lannes–Zarati homomorphism ☆

Sur l'annulation de l'homomorphisme de Lannes–Zarati

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A B S T R A C T

The conjecture on spherical classes states that the Hopf invariant one and the Kervaire invariant one classes are the only elements in $H_*(Q_0S^0)$ belonging to the image of the Hurewicz homomorphism. The Lannes–Zarati homomorphism is a map that corresponds to an associated graded (with a certain filtration) of the Hurewicz map. The algebraic version of the conjecture predicts that the s-th Lannes–Zarati homomorphism vanishes in any positive stems for $s > 2$. In the article, we prove the conjecture for the fifth Lannes–Zarati homomorphism.

R É S U M É

La conjecture sur les classes sphériques affirme que les classes détectées par l'invariant de Hopf et l'invariant de Kervaire sont les seules dans $H_*(Q_0S^0)$ dans l'image de l'homomorphisme de Hurewicz. L'homomorphisme de Lannes–Zarati est l'application correspondant au gradué (pour une certaine filtration) de l'homomorphisme de Hurewicz. La version algébrique de la conjecture prédit que le s-ième homomorphisme de Lannes–Zarati s'annule en tout degré positif pour $s > 2$. Dans cette note, nous démontrons la conjecture pour le cinquième homomorphisme de Lannes–Zarati.

1. Introduction and statement of results

Let $H : \pi_*^s(S^0) \cong \pi_*(Q_0S^0) \to H_*(Q_0S^0)$ be the Hurewicz homomorphism of the basepoint component Q_0S^0 in the infinite loop space $QS^0 = \lim_n \Omega^n S^n$. Here homology is taken with coefficients in $\mathbb{F}_2$, the field of two elements. The long-standing conjecture on spherical classes reads as follows.

Conjecture 1.1. *The Hopf invariant one and the Kervaire invariant one classes are the only elements detected by the Hurewicz homomorphism.*

(See Curtis [5], Snaith and Tornehave [18], and Wellington [19] for a discussion.)

An algebraic version of this problem goes as follows. Let $P_s = \mathbb{F}_2[x_1, \dots, x_s]$ be the polynomial algebra on s indeterminates $x_1, \dots, x_s$, each of degree 1. Let the general linear group $GL_s = GL(s, \mathbb{F}_2)$ and the mod 2 Steenrod algebra $\mathcal{A}$ both act

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on P_s in the usual way. The Dickson algebra is the algebra of invariants, $D_s := \mathbb{F}_2[x_1, \dots, x_s]^{GL_s}$, which inherits a structure of module over the Steenrod algebra from P_s. In [16], Lannes and Zarati constructed a homomorphism:

$$\varphi_s : \mathrm{Ext}_{\mathcal{A}}^{s,s+d}(\mathbb{F}_2, \mathbb{F}_2) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} D_s)_d^*,$$

which corresponds to an associated graded of the Hurewicz map. The proof of this assertion was sketched by Lannes [15] and by Goerss [8]. The Hopf invariant one and the Kervaire invariant one classes are respectively represented by certain permanent cycles in $\mathrm{Ext}_{\mathcal{A}}^{1,*}(\mathbb{F}_2, \mathbb{F}_2)$ and $\mathrm{Ext}_{\mathcal{A}}^{2,*}(\mathbb{F}_2, \mathbb{F}_2)$, on which φ_1 and φ_2 are non-zero (see Adams [1], Browder [3], Lannes–Zarati [16]). Therefore, we are led to an algebraic version of the classical conjecture on spherical classes as follows.

Conjecture 1.2. *(See [9].) $\varphi_s = 0$ in any positive stems, for $s > 2$.*

We now summarize Singer's invariant-theoretic description of the lambda algebra [17]. According to Dickson [6], one has $D_s \cong \mathbb{F}_2[Q_{s,s-1}, \dots, Q_{s,0}]$, where $Q_{s,i}$ denotes the Dickson invariant of degree $2^s - 2^i$. Singer set $\Gamma_s = D_s[Q_{s,0}^{-1}]$, the localization of D_s given by inverting $Q_{s,0}$, and defined Γ_s^+ to be a certain "not too large" submodule of Γ_s. He also equipped $\Gamma^+ = \bigoplus_s \Gamma_s^+$ with a differential $\partial : \Gamma_s^+ \to \Gamma_{s-1}^+$ and a coproduct. Then, he showed that the differential coalgebra Γ^+ is dual to the lambda algebra of the six authors of [2]. Thus, $H_s(\Gamma^+) \cong \mathrm{Tor}_s^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$.

Theorem 1.3. *(See [11].) The inclusion $D_s \subset \Gamma_s^+$ is a chain-level representation of the Lannes–Zarati dual homomorphism, $\varphi_s^* : \mathbb{F}_2 \otimes_{\mathcal{A}} D_s \to \mathrm{Tor}_s^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$.*

Conjecture 1.2 was established for $s = 3$ and 4 respectively in [10] and [12]. That φ_s vanishes for $s > 2$ on the decomposable elements in $\mathrm{Ext}_{\mathcal{A}}^s(\mathbb{F}_2, \mathbb{F}_2)$ and on the Singer transfer's image was respectively proved in [14] and [13].

The goal of this article is to prove the following.

Theorem 1.4. *The fifth Lannes–Zarati homomorphism, $\varphi_5 : \mathrm{Ext}_{\mathcal{A}}^{5,5+d}(\mathbb{F}_2, \mathbb{F}_2) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} D_5)_d^*$, vanishes in any positive stems.*

2. Proof of the theorem

Let Λ be the (opposite) lambda algebra, in which the product in lambda symbols is written in the order opposite to that used in [2]. (See Singer [17, p. 687] for a precise definition of Λ.) The lambda algebra is dual to Singer's coalgebra Γ^+.

Theorem 2.1. *(See [4, Thm. 1.3].) The following classes form an $\mathbb{F}_2$-basis for the indecomposable elements in $\mathrm{Ext}_{\mathcal{A}}^5(\mathbb{F}_2, \mathbb{F}_2)$:*

(1) $Ph_1 = [\lambda_7\lambda_0^3\lambda_2 + (\lambda_1^2\lambda_4\lambda_2 + \lambda_1\lambda_4\lambda_2\lambda_1 + \lambda_4\lambda_1^2\lambda_2)\lambda_1] \in \mathrm{Ext}_{\mathcal{A}}^{5,14}(\mathbb{F}_2, \mathbb{F}_2)$,
(2) $Ph_2 = [\lambda_7\lambda_0^3\lambda_4 + (\lambda_1^3\lambda_5 + \lambda_1^2\lambda_4\lambda_2 + \lambda_1\lambda_4\lambda_2\lambda_1 + \lambda_4\lambda_1^2\lambda_2)\lambda_3 + \lambda_7\lambda_0^2\lambda_2\lambda_2 + \lambda_7\lambda_0\lambda_2\lambda_1\lambda_1] \in \mathrm{Ext}_{\mathcal{A}}^{5,16}(\mathbb{F}_2, \mathbb{F}_2)$,
(3) $n_i = [(Sq^0)^i(\lambda_7^2\lambda_5\lambda_3\lambda_9 + \lambda_7\lambda_{15}\lambda_3\lambda_0\lambda_6 + \lambda_7\lambda_{15}\lambda_1\lambda_5\lambda_3)] \in \mathrm{Ext}_{\mathcal{A}}^{5,2^{i+5}+2^{i+2}}(\mathbb{F}_2, \mathbb{F}_2)$ *for* $i \geqslant 0$,
(4) $x_i = [(Sq^0)^i(\lambda_{15}\lambda_3^2\lambda_2\lambda_{14} + \lambda_7^2\lambda_4\lambda_{12} + \lambda_7^3\lambda_8^2 + \lambda_{23}\lambda_3^2\lambda_2\lambda_6 + \lambda_{23}\lambda_3^2\lambda_4^2 + \lambda_{15}^2\lambda_1\lambda_4\lambda_2)] \in \mathrm{Ext}_{\mathcal{A}}^{5,2^{i+5}+2^{i+3}+2^{i+1}}(\mathbb{F}_2, \mathbb{F}_2)$ *for* $i \geqslant 0$,
(5) $D_1(i) = [(Sq^0)^i(\lambda_{15}^2\lambda_{11}\lambda_7\lambda_4)] \in \mathrm{Ext}_{\mathcal{A}}^{5,2^{i+5}+2^{i+4}+2^{i+3}+2^i}(\mathbb{F}_2, \mathbb{F}_2)$ *for* $i \geqslant 0$,
(6) $H_1(i) = [(Sq^0)^i(\lambda_{15}^2\lambda_{11}\lambda_7\lambda_{14} + \lambda_{15}^2\lambda_{11}^2\lambda_{10} + \lambda_{15}\lambda_{31}\lambda_7\lambda_1\lambda_8 + \lambda_{15}\lambda_{31}\lambda_3\lambda_7\lambda_6 + \lambda_{15}\lambda_{31}\lambda_7\lambda_5\lambda_4)] \in \mathrm{Ext}_{\mathcal{A}}^{5,2^{i+6}+2^{i+1}+2^i}(\mathbb{F}_2, \mathbb{F}_2)$ *for* $i \geqslant 0$,
(7) $Q_3(i) = [(Sq^0)^i(\lambda_{47}(\lambda_7^2\lambda_0\lambda_6 + (\lambda_3^2\lambda_9 + \lambda_9\lambda_3^2)\lambda_5 + \lambda_3\lambda_9\lambda_5\lambda_3 + \lambda_3^2\lambda_{11}\lambda_3) + \lambda_{15}^2\lambda_{11}\lambda_{23}\lambda_3)] \in \mathrm{Ext}_{\mathcal{A}}^{5,2^{i+6}+2^{i+3}}(\mathbb{F}_2, \mathbb{F}_2)$ *for* $i \geqslant 0$,
(8) $K_i = [(Sq^0)^i(\lambda_{63}\lambda_{15}\lambda_{47}\lambda_0^2 + \lambda_{63}\lambda_{47}(\lambda_3^2\lambda_9 + \lambda_9\lambda_3^2) + \lambda_{31}^2\lambda_{11}\lambda_{21})] \in \mathrm{Ext}_{\mathcal{A}}^{5,2^{i+7}+2^{i+1}}(\mathbb{F}_2, \mathbb{F}_2)$ *for* $i \geqslant 0$,
(9) $J_i = [(Sq^0)^i(\lambda_{95}(\lambda_7^2\lambda_{19} + \lambda_{19}\lambda_7^2)\lambda_0 + \lambda_{31}^2\lambda_{23}\lambda_{43}\lambda_0 + \lambda_{63}\lambda_{15}\lambda_{31}\lambda_{19}\lambda_0)] \in \mathrm{Ext}_{\mathcal{A}}^{5,2^{i+7}+2^{i+2}+2^i}(\mathbb{F}_2, \mathbb{F}_2)$ *for* $i \geqslant 0$,
(10) $T_i = [(Sq^0)^i((\lambda_{31}^2\lambda_{79} + \lambda_{79}\lambda_{31}^2)\lambda_0^2 + \lambda_{63}^2(\lambda_3^2\lambda_9 + \lambda_9\lambda_3^2))] \in \mathrm{Ext}_{\mathcal{A}}^{5,2^{i+7}+2^{i+4}+2^{i+1}}(\mathbb{F}_2, \mathbb{F}_2)$ *for* $i \geqslant 0$,
(11) $V_i = [(Sq^0)^i(\lambda_{63}\lambda_{15}\lambda_{47}\lambda_{31}\lambda_0)] \in \mathrm{Ext}_{\mathcal{A}}^{5,2^{i+7}+2^{i+5}+2^i}(\mathbb{F}_2, \mathbb{F}_2)$ *for* $i \geqslant 0$,
(12) $V_i' = [(Sq^0)^i(\lambda_{191}\lambda_{31}\lambda_7\lambda_{23}\lambda_0 + \lambda_{63}\lambda_{127}\lambda_{15}\lambda_{47}\lambda_0)] \in \mathrm{Ext}_{\mathcal{A}}^{5,2^{i+8}+2^i}(\mathbb{F}_2, \mathbb{F}_2)$ *for* $i \geqslant 0$,
(13) $U_i = [(Sq^0)^i(\lambda_{191}(\lambda_{15}^2\lambda_{39} + \lambda_{39}\lambda_{15}^2)\lambda_0 + \lambda_{63}^2\lambda_{47}\lambda_{87}\lambda_0 + \lambda_{127}\lambda_{31}\lambda_{63}\lambda_{39}\lambda_0)] \in \mathrm{Ext}_{\mathcal{A}}^{5,2^{i+8}+2^{i+3}+2^i}(\mathbb{F}_2, \mathbb{F}_2)$ *for* $i \geqslant 0$.

Let $Q(i_0, \dots, i_4)$ denote $Q_{5,0}^{i_0} Q_{5,1}^{i_1} Q_{5,2}^{i_2} Q_{5,3}^{i_3} Q_{5,4}^{i_4}$ for abbreviation.

Theorem 2.2. *(See [7, Thm. 5.7].) The following elements form a basis for the $\mathbb{F}_2$-vector space $\mathbb{F}_2 \otimes_{\mathcal{A}} D_5$:*

(1) $Q(0, 0, 0, 0, 2^{d-1})$ *for* $1 \leqslant d$,
(2) $Q(0, 0, 1, 2^c - 1, 2^{d+1} - 2^c - 1)$ *for* $1 \leqslant c \leqslant d$,

(3) $Q(0, 2, 2^b - 1, 2^{c+1} - 2^b - 1, 2^c - 1)$ *for* $2 \leqslant b \leqslant c$,
(4) $Q(0, 2, 2^b - 1, 2^{c+1} - 2^b - 1, 2^{d+1} - 2^{c+1} - 1)$ *for* $2 \leqslant b \leqslant c < d$,
(5) $Q(1, 1, 2^b - 1, 2^{c+1} - 2^b - 1, 2^c - 1)$ *for* $1 \leqslant b \leqslant c$,
(6) $Q(1, 1, 2^b - 1, 2^{c+1} - 2^b - 1, 2^{d+1} - 2^{c+1} - 1)$ *for* $1 \leqslant b \leqslant c < d$,
(7) $Q(3, 2^a - 1, 2^{b+1} - 2^a - 1, 2^{c+1} - 2^{b+1} - 1, 2^{d+1} - 2^{c+1} - 1)$ *for* $2 \leqslant a \leqslant b < c < d$,
(8) $Q(3, 2^a - 1, 2^{b+1} - 2^a - 1, 2^{c+1} - 2^{b+1} - 1, 2^c - 1)$ *for* $2 \leqslant a \leqslant b < c$,
(9) $Q(3, 2^a - 1, 2^{b+1} - 2^a - 1, 2^b - 1, 2^{d+2} - 3.2^b - 1)$ *for* $2 \leqslant a \leqslant b \leqslant d$.

They are respectively of the following degrees: (1) $2^{d+4} - 16$, (2) $2^{d+5} + 2^{c+3} - 12$, (3) $2^{b+2} + 2^{c+6} - 8$, (4) $2^{b+2} + 2^{c+4} + 2^{d+5} - 8$, (5) $2^{b+2} + 2^{c+6} - 7$, (6) $2^{b+2} + 2^{c+4} + 2^{d+5} - 7$, (7) $2^{a+1} + 2^{b+3} + 2^{c+4} + 2^{d+5} - 5$, (8) $2^{a+1} + 2^{b+3} + 2^{c+6} - 5$, *and* (9) $2^{a+1} + 2^{b+5} + 2^{d+6} - 5$.

Proof of Theorem 1.4. In [14], F. Peterson and the first-named author proved that φ_s vanishes on any decomposable elements for $s > 2$ by showing that $\varphi_* = \bigoplus_s \varphi_s$ is a homomorphism of algebras and, more importantly, that the product of the algebra $\bigoplus_s (\mathbb{F}_2 \otimes_{\mathcal{A}} D_s)^*$ is trivial, except for the case $(\mathbb{F}_2 \otimes_{\mathcal{A}} D_1)^* \otimes (\mathbb{F}_2 \otimes_{\mathcal{A}} D_1)^* \to (\mathbb{F}_2 \otimes_{\mathcal{A}} D_2)^*$. Therefore, we need only to show φ_5 vanishing on any indecomposable elements.

According to Chen's Theorem 2.1, the indecomposable generators of $\mathrm{Ext}^5_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ form the 13 Sq^0-families initiated by the following classes:

$$Ph_1, \quad Ph_2, \quad n_0, \quad x_0, \quad D_1(0), \quad H_1(0), \quad Q_3(0), \quad K_0, \quad J_0, \quad T_0, \quad V_0, \quad V'_0, \quad U_0.$$

Let a_0 denote one of the above 13 classes. Furthermore, set $a_i = (Sq^0)^i(a_0)$, for $i \geqslant 0$. From theorem [12, Thm. 3.1], we have:

$$\varphi_5(a_i) = \varphi_5\big(Sq^0\big)^i(a_0) = \big(Sq^0\big)^i \varphi_5(a_0).$$

So, in order to prove that $\varphi_5(a_i) = 0$ for any i, it suffices to show $\varphi_5(a_0) = 0$.

The proof is divided into two steps.

Step 1: Let a_0 be one of the first 12 indecomposable classes in $\mathrm{Ext}^5_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ given above: Ph_1, Ph_2, n_0, x_0, $D_1(0)$, $H_1(0)$, $Q_3(0)$, K_0, J_0, T_0, V_0, V'_0. We show $\varphi_5(a_0) = 0$ by checking that the stem of a_0 is different from degrees of all the generators of $\mathbb{F}_2 \otimes_{\mathcal{A}} D_5$ given by Theorem 2.2. We check this fact case by case. To have a pattern for the routine computation, we give here the record just for one case.

Case $a_0 = T_0$ *of stem* 141. We combine the stem of T_0 with the degree of each of the generators in $\mathbb{F}_2 \otimes_{\mathcal{A}} D_5$:

(1) $2^{d+4} = 16 + 141 = 2^7 + 2^4 + 2^3 + 2^2 + 1$, no solution;
(2) $2^{d+5} + 2^{c+3} = 12 + 141 = 2^7 + 2^4 + 2^3 + 1$, no solution;
(3) $2^{b+2} + 2^{c+6} = 8 + 141 = 2^7 + 2^4 + 2^2 + 1$, no solution;
(4) $2^{b+2} + 2^{c+4} + 2^{d+5} = 8 + 141 = 2^7 + 2^4 + 2^2 + 1$, no solution;
(5) $2^{b+2} + 2^{c+6} = 7 + 141 = 2^7 + 2^4 + 2^2$, no solution;
(6) $2^{b+2} + 2^{c+4} + 2^{d+5} = 7 + 141 = 2^7 + 2^4 + 2^2$, $d = 2$, $c = 0$, $b = 0$, it does not satisfy $b > 0$, no solution;
(7) $2^{a+1} + 2^{b+3} + 2^{c+4} + 2^{d+5} = 5 + 141 = 2^7 + 2^4 + 2$, no solution;
(8) $2^{a+1} + 2^{b+3} + 2^{c+6} = 5 + 141 = 2^7 + 2^4 + 2$, $c = 1$, $b = 1$, $a = 0$, it does not satisfy $b < c$, no solution;
(9) $2^{a+1} + 2^{b+5} + 2^{d+6} = 5 + 141 = 2^7 + 2^4 + 2$, no solution.

Step 2: Consider $a_0 = U_0$ of stem 260. We combine the stem of U_0 with the degree of each of the generators in $\mathbb{F}_2 \otimes_{\mathcal{A}} D_5$: The unique solution is given by the equation:

$$2^{d+5} + 2^{c+3} = 12 + 260 = 2^8 + 2^4 \quad \Longleftrightarrow \quad d = 3, \ c = 1.$$

More precisely, $Q_{5,2}Q_{5,3}Q^{13}_{5,4}$ is the only generator in $\mathbb{F}_2 \otimes_{\mathcal{A}} D_5$, whose degree equals the stem of U_0.

Let $\langle \cdot, \cdot \rangle$ be the usual dual paring $\mathrm{Tor}^{\mathcal{A}}_s \otimes \mathrm{Ext}^s_{\mathcal{A}} \to \mathbb{F}_2$. We then have:

$$\begin{aligned}
&\big\langle \big[Q_{5,2}Q_{5,3}Q^{13}_{5,4}\big], \varphi_5\big[\lambda_{191}\big(\lambda^2_{15}\lambda_{39} + \lambda_{39}\lambda^2_{15}\big)\lambda_0 + \lambda^2_{63}\lambda_{47}\lambda_{87}\lambda_0 + \lambda_{127}\lambda_{31}\lambda_{63}\lambda_{39}\lambda_0\big]\big\rangle \\
&\quad = \big\langle \varphi^*_5\big[Q_{5,2}Q_{5,3}Q^{13}_{5,4}\big], \big[\lambda_{191}\big(\lambda^2_{15}\lambda_{39} + \lambda_{39}\lambda^2_{15}\big)\lambda_0 + \lambda^2_{63}\lambda_{47}\lambda_{87}\lambda_0 + \lambda_{127}\lambda_{31}\lambda_{63}\lambda_{39}\lambda_0\big]\big\rangle \\
&\quad = \big\langle \big[Q_{5,2}Q_{5,3}Q^{13}_{5,4}\big], \big[\lambda_{191}\big(\lambda^2_{15}\lambda_{39} + \lambda_{39}\lambda^2_{15}\big)\lambda_0 + \lambda^2_{63}\lambda_{47}\lambda_{87}\lambda_0 + \lambda_{127}\lambda_{31}\lambda_{63}\lambda_{39}\lambda_0\big]\big\rangle
\end{aligned}$$

by Theorem 1.3. On the other hand, we obviously observe:

$$\begin{aligned}
Q_{5,2} = {} & \big(v_1^{14}v_2^7v_3^4v_4^2 + v_1^{14}v_2^8v_3^3v_4^2 + v_1^{16}v_2^6v_3^3v_4^2 + v_1^{14}v_2^8v_3^4v_4 + v_1^{16}v_2^6v_3^4v_4 + v_1^{16}v_2^8v_3^2v_4\big)v_5 \\
& + v_1^{14}v_2^8v_3^4v_4^2 + v_1^{16}v_2^6v_3^4v_4^2 + v_1^{16}v_2^8v_3^2v_4^2 + v_1^{16}v_2^8v_3^4,
\end{aligned}$$

$$Q_{5,3} = \left(v_1^{12}v_2^6v_3^3v_4^2 + v_1^{12}v_2^6v_3^4v_4 + v_1^{12}v_2^8v_3^2v_4 + v_1^{16}v_2^4v_3^2v_4\right)v_5 + v_1^{12}v_2^6v_3^4v_4^2$$
$$+ v_1^{12}v_2^8v_3^2v_4^2 + v_1^{16}v_2^4v_3^2v_4^2 + v_1^{12}v_2^8v_3^4 + v_1^{16}v_2^4v_3^4 + v_1^{16}v_2^8,$$
$$Q_{5,4} = v_1^8v_2^4v_3^2v_4v_5 + v_1^8v_2^4v_3^2v_4^2 + v_1^8v_2^4v_3^4 + v_1^8v_2^8 + v_1^{16},$$

where $v_1 = V_1$, $v_k = V_k/V_1 \cdots V_{k-1}$ $(k \geqslant 2)$, with $V_k = \prod_{c_j \in \mathbb{F}_2}(c_1x_1 + \cdots + c_{k-1}x_{k-1} + x_k)$. So, all the exponents of v_1 occurring in the expression of $Q_{5,2}Q_{5,3}Q_{5,4}^{13}$ in terms of v_1, v_2, v_3, v_4, v_5 are even. Since the dual pairing $\mathrm{Tor}_s^{\mathcal{A}} \otimes \mathrm{Ext}_{\mathcal{A}}^s \to \mathbb{F}_2$ is induced in homology by the dual pairing $\Gamma_s^+ \otimes \Lambda^s \to \mathbb{F}_2$ that allows us to identify Γ_s^+ with the dual of Λ^s (see [17, Sections 7–8]), we get

$$\left\langle \left[Q_{5,2}Q_{5,3}Q_{5,4}^{13}\right], \left[\lambda_{191}\left(\lambda_{15}^2\lambda_{39} + \lambda_{39}\lambda_{15}^2\right)\lambda_0 + \lambda_{63}^2\lambda_{47}\lambda_{87}\lambda_0 + \lambda_{127}\lambda_{31}\lambda_{63}\lambda_{39}\lambda_0\right]\right\rangle$$
$$= \left\langle Q_{5,2}Q_{5,3}Q_{5,4}^{13}, \lambda_{191}\left(\lambda_{15}^2\lambda_{39} + \lambda_{39}\lambda_{15}^2\right)\lambda_0 + \lambda_{63}^2\lambda_{47}\lambda_{87}\lambda_0 + \lambda_{127}\lambda_{31}\lambda_{63}\lambda_{39}\lambda_0\right\rangle$$
$$= 0.$$

In other words, $\varphi_5(U_0) = 0$.

Combining Step 1 and Step 2, we get a complete proof for the theorem. □

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References

[1] J.F. Adams, On the non-existence of elements of Hopf invariant one, Ann. Math. 72 (1960) 20–104.
[2] A.K. Bousfield, E.B. Curtis, D.M. Kan, D.G. Quillen, D.L. Rector, J.W. Schlesinger, The mod p lower central series and the Adams spectral sequence, Topology 5 (1966) 331–342.
[3] W. Browder, The Kervaire invariant of a framed manifold and its generalization, Ann. Math. 90 (1969) 157–186.
[4] T.W. Chen, Determination of $\mathrm{Ext}_{\mathcal{A}}^{5,*}(\mathbb{Z}/2, \mathbb{Z}/2)$, Topol. Appl. 158 (2011) 660–689.
[5] E.B. Curtis, The Dyer–Lashof algebra and the lambda algebra, Ill. J. Math. 18 (1975) 231–246.
[6] L.E. Dickson, A fundamental system of invariants of the general modular linear group with a solution of the form problem, Trans. Amer. Math. Soc. 12 (1911) 75–98.
[7] V. Giambalvo, F.P. Peterson, A-generators for ideals in the Dickson algebra, J. Pure Appl. Algebra 158 (2001) 161–182.
[8] P.G. Goerss, Unstable projectives and stable Ext: with applications, Proc. Lond. Math. Soc. 53 (1986) 539–561.
[9] Nguyễn H.V. Hưng, Spherical classes and the algebraic transfer, Trans. Amer. Math. Soc. 349 (1997) 3893–3910.
[10] Nguyễn H.V. Hưng, The weak conjecture on spherical classes, Math. Z. 231 (1999) 727–743.
[11] Nguyễn H.V. Hưng, Spherical classes and the lambda algebra, Trans. Amer. Math. Soc. 353 (2001) 4447–4460.
[12] Nguyễn H.V. Hưng, On triviality of Dickson invariants in the homology of the Steenrod algebra, Math. Proc. Camb. Philos. Soc. 134 (2003) 103–113.
[13] Nguyễn H.V. Hưng, Trân N. Nam, The hit problem for the Dickson algebra, Trans. Amer. Math. Soc. 353 (2001) 5029–5040.
[14] Nguyễn H.V. Hưng, F.P. Peterson, Spherical classes and the Dickson algebra, Math. Proc. Camb. Philos. Soc. 124 (1998) 253–264.
[15] J. Lannes, Sur le n-dual du n-ème spectre de Brown–Gitler, Math. Z. 199 (1988) 29–42.
[16] J. Lannes, S. Zarati, Sur les foncteurs dérivés de la déstabilisation, Math. Z. 194 (1987) 25–59.
[17] W.M. Singer, Invariant theory and the lambda algebra, Trans. Amer. Math. Soc. 280 (1983) 673–693.
[18] V. Snaith, J. Tornehave, On $\pi_*^S(BO)$ and the Arf invariant of framed manifolds, Contemp. Math. 12 (1982) 299–313.
[19] R.J. Wellington, The unstable Adams spectral sequence of free iterated loop spaces, Mem. Amer. Math. Soc. 258 (1982).