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A q-analogue for bi^{s}nomial coefficients and generalized Fibonacci sequences

Un q-analogue pour les coefficients bi^{s}nomiaux et les suites de Fibonacci généralisées

Hacène Belbachir [a], Athmane Benmezai [b,c]

[a] *USTHB, Faculty of Mathematics, RECITS Laboratory, DG-RSDT, BP 32, El Alia 16111, Bab Ezzouar, Algiers, Algeria*
[b] *University of Dely Brahim, Fac. of Eco. & Manag. Sc., RECITS Laboratory, DG-RSDT, Rue Ahmed Ouaked, Dely Brahim, Algiers, Algeria*
[c] *University of Oran, Faculty of Sciences, BP 1524, ELM Naouer, 31000, Oran, Algeria*

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ABSTRACT

A new q-analogue of bi^{s}nomial coefficients is proposed according to the generalized q-Fibonacci sequence suggested by Cigler's approach.

RÉSUMÉ

Nous proposons une nouvelle variante de q-analogue pour les coefficients binomiaux généralisés appelés coefficients bi^{s}nomiaux. Elle est basée sur les suites q-Fibonacci proposées par Cigler.

1. Introduction

For $s \geqslant 1$, bi^{s}nomial coefficients denoted by $\binom{n}{k}_s$ are considered as extensions of binomial coefficients $\binom{n}{k}$ and are obtained by the multinomial expansion:

$$\left(1+x+x^2+\cdots+x^s\right)^n=\sum_{k\geqslant 0}\binom{n}{k}_s x^k, \tag{1}$$

where $\binom{n}{k}_1=\binom{n}{k}$ is the classical binomial coefficient, and for $k>ns$ or $k<0$, $\binom{n}{k}_s=0$.

They satisfy the following recursion:

$$\binom{n}{k}_s=\sum_{j=0}^{n}\binom{n}{j}\binom{j}{k-j}_{s-1}. \tag{2}$$

For an appropriate introduction of this numbers, see Andrews and Baxter [1], Belbachir et *al.* [6], Bollinger [7] and Smith and Hogatt [9].

E-mail addresses: hbelbachir@usthb.dz, hacenebelbachir@gmail.com (H. Belbachir), athmanebenmezai@gmail.com (A. Benmezai).

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Using the classical binomial and multinomial coefficients (see [2,6]), we obtain the following formulae:

$$\binom{n}{k}_s = \sum_{j_1+j_2+\cdots+j_s=k} \binom{n}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-1}}{j_s}, \tag{3}$$

$$\binom{n}{k}_s = \sum_{j_1+2j_2+\cdots+sj_s=k} \binom{n}{j_1, j_2, \ldots, j_s}, \tag{4}$$

$$\binom{n}{k}_s = \sum_{j=0}^{\lfloor k/(s+1)\rfloor} (-1)^j \binom{n}{j}\binom{n-1+k-j(s+1)}{n-1}. \tag{5}$$

The following recursion is also satisfied:

$$\binom{n}{k}_s = \sum_{j=0}^{s} \binom{n-1}{k-j}_s.$$

The coefficients $q^{\binom{k}{2}}\left[{n \atop k}\right]$ are considered as a q-analogue of binomial coefficients. They appear as the coefficients of the binomial expansion of x, y such that $yx = qxy$:

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} y^k,$$

or as the coefficients of the following product:

$$(1+z)(1+qz)\bigl(1+q^2z\bigr)\cdots\bigl(1+q^{n-1}z\bigr) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} z^k. \tag{6}$$

2. A q-analogue of bi^{s}nomial coefficients

For $a = \mathrm{e}^{\mathrm{i}\frac{2\pi}{s+1}}$, the multinomial expansion gives:

$$\begin{aligned}\left(\sum_{j=0}^{s} x^j\right)^n &= \left(\prod_{r=1}^{s}\bigl(x-a^r\bigr)\right)^n \\ &= \prod_{r=1}^{s}\left(\sum_{j=0}^{n}\binom{n}{j} x^j \bigl(-a^r\bigr)^{n-j}\right) \\ &= \sum_{k=0}^{ns}\left(\sum_{j_1+j_2+\cdots+j_s=k}\binom{n}{j_1}\binom{n}{j_2}\cdots\binom{n}{j_s}(-1)^{ns-k} a^{\sum_{r=1}^{s} r(n-j_r)}\right)x^k.\end{aligned}$$

By identification, we obtain a new expression of the bi^{s}nomial coefficients:

Theorem 2.1. *The following identity holds*:

$$\binom{n}{k}_s = \sum_{j_1+j_2+\cdots+j_s=k} \binom{n}{j_1}\binom{n}{j_2}\cdots\binom{n}{j_s}(-1)^k a^{-\sum_{r=1}^{s} rj_r}. \tag{7}$$

Now, we are able to propose a definition of the q-analogue of the bi^{s}nomial coefficients.

Definition 2.1. We define a q-analogue of bi^{s}nomial coefficients by:

$$\begin{bmatrix} n \\ k \end{bmatrix}_s = \sum_{j_1+j_2+\cdots+j_s=k} \begin{bmatrix} n \\ j_1 \end{bmatrix}_1 \begin{bmatrix} n \\ j_2 \end{bmatrix}_1 \cdots \begin{bmatrix} n \\ j_s \end{bmatrix}_1 (-1)^k a^{-\sum_{r=1}^{s} rj_r}. \tag{8}$$

where $\left[{n \atop j}\right]_1 = q^{\binom{j}{2}}\left[{n \atop j}\right]$.

Remark 2.2. For $s=1$, we obtain the classical q-binomial coefficients $q^{\binom{k}{2}}\left[\begin{smallmatrix} n\\ k\end{smallmatrix}\right]$. We use the notation $\left[\begin{smallmatrix} n\\ k\end{smallmatrix}\right]_1$ to indicate the term $q^{\binom{k}{2}}\left[\begin{smallmatrix} n\\ k\end{smallmatrix}\right]$.

According to relation (7), these coefficients seem to be natural as the classical q-analogues of bi^{s}nomial coefficients. Moreover, the product $\prod_{j=0}^{n-1}(1+q^jz+\cdots+(q^jz)^s)$ appears as a natural q-analogue of multinomial expansion; we will show below that $\left[\begin{smallmatrix} n\\ k\end{smallmatrix}\right]_s$ is the coefficient of the k-th term of the product.

Theorem 2.3. *We have*:

$$\prod_{j=0}^{n-1}\left(1+q^jz+\cdots+\left(q^jz\right)^s\right)=\sum_{k=0}^{ns}\begin{bmatrix} n\\ k\end{bmatrix}_s z^k. \tag{9}$$

Proof. Knowing that $a=\mathrm{e}^{\mathrm{i}\frac{2\pi}{s+1}}$, we have $(1+z+z^2+\cdots+z^s)=\prod_{j=1}^{s}(z-a^j)$. Hence,

$$\begin{aligned}\prod_{j=0}^{n-1}\left(1+q^jz+\cdots+\left(q^jz\right)^s\right)&=\prod_{r=1}^{s}\left(z-a^r\right)\left(qz-a^r\right)\cdots\left(q^{n-1}z-a^r\right)\\&=\prod_{r=1}^{s}\left(\sum_{k=0}^{n}\begin{bmatrix} n\\ j\end{bmatrix}q^{\binom{j}{2}}z^j\left(-a^r\right)^{n-j}\right)\\&=\sum_{k=0}^{ns}\left(\sum_{j_1+\cdots+j_s=k}\begin{bmatrix} n\\ j_1\end{bmatrix}\cdots\begin{bmatrix} n\\ j_s\end{bmatrix}q^{\sum_{r=1}^{s}\binom{j_r}{2}}(-1)^{ns-k}a^{\sum_{r=1}^{s}r(n-j_r)}\right)z^k\\&=\sum_{k=0}^{ns}\begin{bmatrix} n\\ k\end{bmatrix}_s z^k.\end{aligned}$$

The last line is deduced from the previous one, by: $a^{\sum_{r=1}^{s}rn}=(-1)^{sn}$. □

Remark 2.4. Our definition for the q-analogue of bi^{s}nomial coefficients and the definition suggested by Andrews and Baxter are two different approaches (see for example the case $s=2$). Each one has its advantages.

The q-binomial coefficients $\left[\begin{smallmatrix} n\\ k\end{smallmatrix}\right]_1$ satisfy two recursions:

$$\begin{bmatrix} n\\ k\end{bmatrix}_1=q^k\begin{bmatrix} n-1\\ k\end{bmatrix}_1+q^{k-1}\begin{bmatrix} n-1\\ k-1\end{bmatrix}_1, \tag{10}$$

$$\begin{bmatrix} n\\ k\end{bmatrix}_1=\begin{bmatrix} n-1\\ k\end{bmatrix}_1+q^{n-1}\begin{bmatrix} n-1\\ k-1\end{bmatrix}_1. \tag{11}$$

Using the two relations:

$$\prod_{j=0}^{n-1}\left(1+q^jz+\cdots+\left(q^jz\right)^s\right)=\left(1+z+\cdots+z^s\right)\prod_{j=0}^{n-2}\left(1+q^{j+1}z+\cdots+\left(q^{j+1}z\right)^s\right),$$

$$\prod_{j=0}^{n-1}\left(1+q^jz+\cdots+\left(q^jz\right)^s\right)=\left(1+q^{n-1}z+\cdots+\left(q^{n-1}z\right)^s\right)\prod_{j=0}^{n-2}\left(1+q^jz+\cdots+\left(q^jz\right)^s\right),$$

we establish the following theorem, which generalizes the recursions (10) and (11) to the q-bi^{s}nomial coefficients.

Theorem 2.5. *The q-analogue of bi^{s}nomial coefficients satisfy the two following recursions:*

$$\begin{bmatrix} n\\ k\end{bmatrix}_s=\sum_{j=0}^{s}q^{k-j}\begin{bmatrix} n-1\\ k-j\end{bmatrix}_s, \tag{12}$$

$$\begin{bmatrix} n\\ k\end{bmatrix}_s=\sum_{j=0}^{s}q^{(n-1)j}\begin{bmatrix} n-1\\ k-j\end{bmatrix}_s. \tag{13}$$

These two recursions are q-analogue of the generalized Pascal formula satisfied by the bi^{s}nomial coefficients:

$$\binom{n}{k}_s = \sum_{j=0}^{s} \binom{n-1}{k-j}_s.$$

3. Generalized Fibonacci sequences and q-analogue

Belbachir and Bencherif [3] consider the sequences the sequences $(\Phi_n^{(s)})_{n \geqslant -s}$ defined by:

$$\begin{cases} (\Phi_0^{(s)}, \Phi_{-1}^{(s)}, \ldots, \Phi_{-s}^{(s)}) = (1, 0, 0, \ldots, 0), \\ \Phi_n^{(s)} = \Phi_{n-1}^{(s)} + \Phi_{n-2}^{(s)} + \cdots + \Phi_{n-s-1}^{(s)} \quad (n \geqslant 1), \end{cases}$$

as the generalized Fibonacci sequences, and in [6], Belbachir et *al.* establish the following:

$$\Phi_n^{(s)} = \sum_{k=0}^{sm-r} \binom{n-k}{k}_s,$$

where m is given by the extended Euclidean algorithm for division: $0 \leqslant r \leqslant s$ and $m(s+1) - r = n$. As the q-analogue of the generalized Fibonacci sequences $(\Phi_n^{(s)})_{n \geqslant -s}$, we introduce the following.

Definition 3.1. For $s \geqslant 1$, we define the q-analogue of the generalized Fibonacci polynomials by:

$$\mathbf{F}_{n+1}^{(s)}(z) = \sum_{k=0}^{sm-r} q^k \begin{bmatrix} n-k \\ k \end{bmatrix}_s z^k,$$

where $0 \leqslant r \leqslant s$ and $m(s+1) - r = n$.

Theorem 3.1. *The generalized q-Fibonacci polynomials satisfy the two following recursions*:

$$\mathbf{F}_{n+1}^{(s)}(z) = \sum_{j=0}^{s} (q^{n-j} z)^j \mathbf{F}_{n-j}^{(s)}\left(\frac{z}{q^j}\right), \tag{14}$$

$$\mathbf{F}_{n+1}^{(s)}\left(\frac{z}{q}\right) = \sum_{j=0}^{s} z^j \mathbf{F}_{n-j}^{(s)}(z), \tag{15}$$

with the initials:

$$\left(\mathbf{F}_1^{(s)}(z), \mathbf{F}_0^{(s)}(z), \mathbf{F}_{-1}^{(s)}(z), \ldots, \mathbf{F}_{-s+1}^{(s)}(z)\right) = (1, 0, 0, \ldots, 0).$$

Proof. For $n < 0$, the term $\mathbf{F}_{n+1}^{(s)}(z)$ is computed over an empty summation index which gives $\mathbf{F}_{n+1}^{(s)}(z) = 0$; otherwise, we have:

$$\begin{aligned} \sum_{j=0}^{s} (q^{n-j} z)^j \mathbf{F}_{n-j}^{(s)}\left(\frac{z}{q^j}\right) &= \sum_{j=0}^{s} (q^{n-j} z)^j \left(\sum_{k \geqslant 0} q^k \begin{bmatrix} n-1-j-k \\ k \end{bmatrix}_s \left(\frac{z}{q^j}\right)^k \right) \\ &= \sum_{j=0}^{s} \left(\sum_{k \geqslant 0} q^{(n-j-k)j+k} \begin{bmatrix} n-1-j-k \\ k \end{bmatrix}_s z^{k+j} \right) \\ &= \sum_{j=0}^{s} \left(\sum_{k \geqslant 0} q^{(n-k)j+k-j} \begin{bmatrix} n-1-k \\ k-j \end{bmatrix}_s z^k \right) \\ &= \sum_{k \geqslant 0} q^k \left(\sum_{j=0}^{s} q^{(n-k-1)j} \begin{bmatrix} n-1-k \\ k-j \end{bmatrix}_p \right) z^k. \end{aligned}$$

According to (13), we obtain:

$$\sum_{j=0}^{s} (q^{n-j} z)^j \mathbf{F}_{n-j}^{(s)}\left(\frac{z}{q^j}\right) = \sum_{k \geqslant 0} q^k \begin{bmatrix} n-k \\ k \end{bmatrix}_s z^k = \mathbf{F}_{n+1}^{(s)}(z).$$

Also, we have:

$$\sum_{j=0}^{s} z^j \mathbf{F}_{n-j}^{(s)}(z) = \sum_{j=0}^{s} z^j \left(\sum_{k\geqslant 0} q^k \begin{bmatrix} n-1-j-k \\ k \end{bmatrix}_s z^k \right) = \sum_{j=0}^{s} \left(\sum_{k\geqslant 0} q^k \begin{bmatrix} n-1-j-k \\ k \end{bmatrix}_s z^{k+j} \right) = \sum_{k\geqslant 0} \left(\sum_{j=0}^{s} q^{k-j} \begin{bmatrix} n-1-k \\ k-j \end{bmatrix}_s \right) z^k.$$

According to (12), we obtain:

$$\sum_{j=0}^{s} z^j \mathbf{F}_{n-j}^{(s)}(z) = \sum_{k\geqslant 0} \begin{bmatrix} n-k \\ k \end{bmatrix}_s z^k = \mathbf{F}_{n+1}^{(s)}\left(\frac{z}{q}\right). \qquad \square$$

For $s = 1$, we recover the q-Fibonacci polynomials suggested by J. Cigler [8], see also [4,5]

$$\mathbf{F}_{n+1}^{(1)}(z) = \mathbf{F}_{n+1}(z) = \sum_{k\geqslant 0} q^k \begin{bmatrix} n-k \\ k \end{bmatrix}_1 z^k,$$

which satisfy the two following recursions:

$$\mathbf{F}_{n+1}(z) = \mathbf{F}_n(z) + q^{n-1} z \mathbf{F}_{n-1}\left(\frac{z}{q}\right), \tag{16}$$

$$\mathbf{F}_{n+1}\left(\frac{z}{q}\right) = \mathbf{F}_n(z) + z\mathbf{F}_{n-1}(z). \tag{17}$$

The equalities (16) and (17) are respectively a specialization ($s = 1$) of the recursions (14) and (15) satisfied by the q-analogue of the generalized Fibonacci polynomials.

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