C. R. Acad. Sci. Paris, Ser. I 352 (2014) 183–187

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Partial differential equations/Mathematical problems in mechanics

The 3D motion of a solid with large deformations

Mouvement en dimension 3 d'un solide en grandes déformations

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ARTICLE INFO

Article history:
Received 25 October 2013
Accepted 16 January 2014
Available online 3 February 2014

Presented by Philippe G. Ciarlet

ABSTRACT

We study in dimension 3 the motion of a solid with large deformations. The solid may be loaded on its surface by needles, rods, beams, shells, etc. Therefore, it is wise to choose a third gradient theory for the body. It is known that the stretch matrix of the polar decomposition has to be symmetric. This is an internal constraint, which introduces a reaction stress in the Piola–Kirchhoff–Boussinesq stress. We prove that there exists a motion that satisfies the complete equations of Mechanics in a convenient variational framework. This motion is local-in-time for it may be interrupted by a crushing, which entails a discontinuity of velocity with respect to time, i.e., an internal collision.

RÉSUMÉ

On étudie en dimension 3 le mouvement d'un solide en grandes déformations. Ce solide peut être chargé par des pointes, des fils, des poutres, des coques... Cela nous conduit à retenir une théorie du troisième gradient dans le solide. La matrice d'élongation qui apparaît dans la décomposition polaire doit être symétrique. Cette liaison interne introduit une contrainte de réaction qui contribue à la contrainte de Piola–Kirchhoff–Boussinesq. On montre alors qu'il existe un mouvement qui satisfait toutes les équations de la Mécanique dans un cadre variationnel convenable. Ce mouvement est local en temps, car il peut être interrompu par un écrasement provoquant une discontinuité de vitesse par rapport au temps, c'est-à-dire une collision interne.

Version française abrégée

Dans une note précédente, [1], nous avons étudié en dimension 2 le mouvement d'un solide en grandes déformations. Nous étendons les résultats à la dimension 3. On se concentre sur les éléments qui changent ou sont nouveaux :

(i) la condition de non-interpénétration locale ;
(ii) la conservation du moment cinétique, qui donne la matrice antisymétrique de réaction **A**.

Les efforts extérieurs peuvent être appliqués au solide sur sa surface par des plaques ou des coques. Les vitesses des plaques ou coques sont alors égales aux traces sur la surface des vitesses du solide. Comme le principe des puissances

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1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2014.01.007

virtuelles pour les plaques ou coques requiert le second gradient, ici le second gradient de la trace des vitesses du solide, il est raisonnable, pour définir cette trace, d'avoir une théorie du troisième gradient dans le solide lui-même. Le cerclage de piliers nous paraît exemplaire de ce point de vue. Plus généralement, le solide peut être chargé par des pointes, des barres curvilignes, des membranes, des poutres curvilignes et des coques qui nécessitent des théories du zéro, premier et second gradient sur la surface du solide.

La condition de non-interpénétration locale impose à la matrice d'élongation $\mathbf{W}$ d'être dans l'ensemble $C_\alpha \cap \mathcal{S}$ des matrices symétriques dont la somme des valeurs propres est supérieure ou égale à 3α, dont la somme des produits deux à deux des valeurs propres est supérieure ou égale à $3\alpha^2$ et dont le produit des valeurs propres est supérieur ou égal à α^3, avec $1 > \alpha > 0$ (voir [1]). Mécaniquement, on ne peut trop écraser le matériau ; cependant, les raccourcissements peuvent être proches de 100%. La symétrie de la matrice d'élongation $\mathbf{W}$ est une liaison interne qui introduit une réaction, une contrainte antisymétrique $\mathbf{A}$ qui apporte une contribution à la contrainte de Piola–Kirchhoff–Boussinesq $\boldsymbol{\Pi}$ (voir [3]). Les lois de comportement (11) comportent, outre les efforts élastiques, des efforts visqueux représentés par la vitesses d'élongation $\dot{\mathbf{W}}$ et le gradient de la vitesse de rotation grad $\boldsymbol{\Omega}$.

Pour prédire le mouvement du solide, il faut déterminer $\Phi(\mathbf{a}, t)$ et $\mathbf{A}(\mathbf{a}, t)$. Les équations sont la conservation de la quantité de mouvement et la conservation du moment cinétique (4) et (5) données par le principe des puissances virtuelles, les relations cinématiques et les lois de comportement, complétées par les conditions à la limite et les conditions initiales. On montre (Théorème 1) qu'il existe un mouvement local en temps $\Phi(\mathbf{a}, t)$ assez lisse, avec $\Phi \in L^\infty(0, T; H^3(\mathcal{D}_a))$, $\mathrm{d}\Phi/\mathrm{d}t \in L^2(0, T; H^1(\mathcal{D}_a)) \cap L^\infty(0, T; L^2(\mathcal{D}_a))$, $\mathrm{d}^2\Phi/\mathrm{d}t^2$ dans l'espace dual de l'espace des vitesses $\mathcal{V}(T)$ défini par (18), et une contrainte de réaction $\mathbf{A}(\mathbf{a}, t)$ dans l'espace dual de l'espace des vitesses de rotation $\mathcal{V}_{rv}(T)$ défini par (20). Comme il se doit, la réaction $\mathbf{A}$ est parfaite, c'est-à-dire qu'elle ne travaille pas dans le mouvement. Les propriétés de cette matrice sont données par l'équation de conservation du moment cinétique (voir paragraphe 4.1). On prouve un résultat de régularité, $\mathrm{d}^2\Phi/\mathrm{d}t^2 \in L^2(0, \hat{T}; \hat{\mathcal{V}}')$, où l'espace des vitesses $\hat{\mathcal{V}}$ est défini par (19). Le mouvement peut ne pas exister sur toute période de temps, car il peut être interrompu par une collision interne qui se produit lorsque le matériau atteint sa limite d'écrasement.

1. Introduction

In the note [1], we proved the existence of 2-D motions of a solid with large deformations. Here, we extend the result to 3-D motions, for which we prove (again) a local-in-time existence result in the case of large deformations. Two important elements are still present:

(i) a reaction stress matrix $\mathbf{A}$ that keeps the stretch matrix $\mathbf{W}$ symmetric. The reaction matrix $\mathbf{A}$ has been introduced for the equilibrium of the solid, see [3,4];
(ii) a volume third gradient theory for the deformation velocities that are needed to equilibrate surface loads.

The theory introduces also a reaction stress to the local impenetrability condition or to crushing. The motion exists locally in time because it may be interrupted by an internal collision due to crushing.

We keep the notations and general setting of [1]. We focus on the elements which are different, namely:

(i) the local impenetrability condition;
(ii) the angular momentum equation giving the reaction matrix $\mathbf{A}$.

2. Description of the motion and the equations of motion

We consider the motion of a solid located in a smooth bounded domain $\mathcal{D}_a \subset \mathbf{R}^3$. As in [1], we assume neither self-collision nor self-contact and neither collision nor contact with obstacles. Then, in the set of 3×3 matrices $\mathcal{M}$, we introduce the convex:

$$C_\alpha = \left\{\mathbf{B} \in \mathcal{M} \;\middle|\; \operatorname{tr}\mathbf{B} \geqslant 3\alpha,\ \operatorname{tr}(\operatorname{cof}\mathbf{B}) \geqslant 3\alpha^2,\ \det\mathbf{B} \geqslant \alpha^3\right\}, \quad 0 < \alpha < 1, \tag{1}$$

where α is the only physical parameter the value of which we choose different from 1. We recall that for any position Φ, which is kinematically admissible, i.e. differentiable with $\det(\operatorname{grad}\Phi) > 0$, there exists a unique symmetric positive definite matrix $\mathbf{W}$ and a rotation matrix $\mathbf{R}$ with $\det\mathbf{R} = 1$, such that $\operatorname{grad}\Phi = \mathbf{RW}$. With this decomposition, the local impenetrability condition is to require for the stretch matrix that:

$$\mathbf{W} \in C_\alpha. \tag{2}$$

Note in particular that the physical constant α quantifies the resistance of the material to crushing. The 3-D solid may be loaded by shells which are in bilateral contact with the body on its surface. The velocities in the shells are equal to the traces on the surface of the body velocities. Principle of virtual power for shells requires the second space derivatives on the surface. Thus, it is convenient to have a third gradient body theory that ensures that the trace of the second gradient is defined on the surface of the solid. Column hooping is an example of such a loading. The velocities of the hoop points are equal to the velocities of the points of the surface of the column they are in contact with.

Note also that the 3-D solid may also be loaded by needles, wires, membranes, curvilinear rods, curvilinear beams, and shells (rods, beams, and plates on a flat surface). The velocities of deformation we choose are:

$$\operatorname{grad}\vec{V},\ \Delta(\operatorname{grad}\vec{V}),\ \hat{\boldsymbol{\Omega}},\ \operatorname{grad}\hat{\boldsymbol{\Omega}}, \tag{3}$$

where $\vec{V}$ and $\hat{\boldsymbol{\Omega}}$ are virtual velocities and virtual angular velocities (Δ is the Laplacian operator).

The equations of motion result from the principle of virtual power. They hold in $\mathcal{D}_a$,

$$\frac{\mathrm{d}\vec{U}}{\mathrm{d}t} - \operatorname{div}\boldsymbol{\Pi} - \operatorname{div}(\Delta\mathbf{Z}) = \mathbf{f}, \tag{4}$$

$$\operatorname{div}\boldsymbol{\Lambda} + \mathbf{M} + \mathbf{M}^e = 0, \tag{5}$$

where $\mathbf{f}$ and $\mathbf{M}^e$ are the body exterior force and torque, $\vec{U}$ is the actual velocity, $\mathbf{M}$ is the moment, $\boldsymbol{\Pi}$ is the Piola–Kirchhoff–Boussinesq stress tensor and $\mathbf{Z}$ stands for a diffusion stress. The principle of virtual power provides also the boundary conditions. We do not detail them, assuming in the sequel that no surface exterior action is applied. The kinematic relationships are the same as in [1].

3. The constitutive laws

The free energy accounts for the impenetrability condition. In particular, this constraint is related to the presence (in the free energy) of a function $\hat{\Psi}(\mathbf{W})$, which is a smooth approximation from the interior of the indicator function of the set C_α in $\mathcal{M}$. Let $I^{\det}(x)$ be a decreasing non-negative smooth approximation of the indicator function of $[\alpha^3, +\infty)$ from the interior, i.e., such that $I^{\det}(x) = +\infty$ if $x \leqslant \alpha^3$ (for instance, $1/(x-\alpha^3)$ for $x > \alpha^3$). Let $I^{\mathrm{cof}}(x)$ be a decreasing non-negative smooth approximation of the indicator function of $[3\alpha^2, +\infty)$ from the interior, i.e., such that $I^{\mathrm{cof}}(x) = +\infty$ if $x \leqslant 3\alpha^2$. Let $I^{\mathrm{tr}}(x)$ be a decreasing non-negative smooth approximation of the indicator function of $[3\alpha, +\infty)$ from the interior, i.e., such that such $I^{\mathrm{tr}}(x) = +\infty$ if $x \leqslant 3\alpha$. Then, the function $\hat{\Psi}$ may be defined by:

$$\mathbf{B} \to \hat{\Psi}(\mathbf{B}) = \begin{cases} I^{\det}(\det\mathbf{B}) + I^{\mathrm{cof}}(\operatorname{tr}(\operatorname{cof}(\mathbf{B}))) + I^{\mathrm{tr}}(\operatorname{tr}\mathbf{B}), & \text{if } \mathbf{B} \in \mathring{C}_\alpha, \\ +\infty, & \text{if } \mathbf{B} \notin \mathring{C}_\alpha. \end{cases} \tag{6}$$

We have the constitutive laws:

$$\mathbf{Z} = \frac{\partial\Psi}{\partial(\operatorname{grad}\Delta\boldsymbol{\Phi})} = \operatorname{grad}\Delta\boldsymbol{\Phi}, \tag{7}$$

$$\boldsymbol{\Pi} = \mathbf{R}(\mathbf{S}+\mathbf{A}), \quad \mathbf{S} \in \mathcal{S},\ \mathbf{A} \in \mathcal{A}, \tag{8}$$

$$\mathbf{S} = \frac{\partial\Psi}{\partial\mathbf{W}}(\mathbf{W}) + \frac{\partial D}{\partial\dot{\mathbf{W}}}(\dot{\mathbf{W}}) = (\mathbf{W}-\mathbf{I}) + \dot{\mathbf{W}} + \frac{\partial\hat{\Psi}}{\partial\mathbf{W}}(\mathbf{W}), \quad \mathbf{A} \in \partial I_{\mathcal{S}}(\mathbf{W}) = \mathcal{A}, \tag{9}$$

$$\mathbf{M} = \boldsymbol{\Pi}\mathbf{F}^T - \mathbf{F}\boldsymbol{\Pi}^T, \tag{10}$$

$$\boldsymbol{\Lambda} = 4\left(\frac{\partial\Psi}{\partial(\|\operatorname{grad}\mathbf{R}\|^2)}\right)(\operatorname{grad}\mathbf{R})\mathbf{R}^T + \frac{\partial D}{\partial(\operatorname{grad}\boldsymbol{\Omega})}(\operatorname{grad}\boldsymbol{\Omega}) = (\operatorname{grad}\mathbf{R})\mathbf{R}^T + \operatorname{grad}\boldsymbol{\Omega}. \tag{11}$$

Stress $\partial\hat{\Psi}/\partial\mathbf{W}$ is the impenetrability reaction, which intervenes to avoid crushing of the material. Reaction stress $\mathbf{A}$ ensures that the stretch matrix $\mathbf{W}$ is symmetric (see [3,4]). The position $\boldsymbol{\Phi}$ and the reaction stress $\mathbf{A}$ are the main unknowns of the problem. We point out that the moment flux $\operatorname{grad}\boldsymbol{\Omega}$ and the stress $\dot{\mathbf{W}}$ are dissipative.

The boundary conditions and initial conditions are the same as in [1]. They are on the boundary:

$$\dot{\boldsymbol{\Phi}} = 0, \quad \operatorname{grad}\dot{\boldsymbol{\Phi}} = 0, \quad \frac{\partial}{\partial N}\operatorname{grad}\dot{\boldsymbol{\Phi}} = 0, \quad \text{on } \Gamma_0, \tag{12}$$

$$\text{no exterior force is applied}, \quad \operatorname{grad}\dot{\boldsymbol{\Phi}} = 0, \quad \text{on } \Gamma_1. \tag{13}$$

Initial conditions are introduced as well,

$$\boldsymbol{\Phi}(\mathbf{a},0) = \mathbf{a}, \qquad \vec{U}(\mathbf{a},0) = \frac{\partial\boldsymbol{\Phi}}{\partial t}(\mathbf{a},0) = 0, \quad \mathbf{a} \in \mathcal{D}_a. \tag{14}$$

4. The predictive theory. Variational formulation and existence result

The equations describing the motion of the solid are the kinematic relationships, the equations of motion and the constitutive laws, plus boundary and initial conditions.

4.1. A property of the angular momentum equation

From the resulting angular momentum equation:

$$\operatorname{div}\big((\operatorname{grad}\mathbf{R})\mathbf{R}^T\big)+\Delta\boldsymbol{\Omega}+\mathbf{R}\{\mathbf{AW}+\mathbf{WA}+\dot{\mathbf{W}}\mathbf{W}-\mathbf{W}\dot{\mathbf{W}})\}\mathbf{R}^T=\mathbf{0},\quad \text{in } \mathcal{D}_a, \tag{15}$$

with the help of a result detailed in [5], we can deduce the existence of a unique reaction stress $\mathbf{A}$ satisfying (15), depending on $\mathbf{W}\in C_\alpha$, $\mathbf{R}$, $\boldsymbol{\Omega}$ and $\dot{\mathbf{W}}$, such that:

$$\mathbf{A}=\frac{1}{i_1 i_2-i_3}\big\{\big(i_1^2-i_2\big)\mathbf{Y}-\big(\mathbf{W}^2\mathbf{Y}+\mathbf{Y}\mathbf{W}^2\big)\big\}, \tag{16}$$

where the i_i are the invariants of matrix $\mathbf{W}$ and

$$\mathbf{Y}=-\mathbf{R}^T\big\{\operatorname{div}\big((\operatorname{grad}\mathbf{R})\mathbf{R}^T\big)+\Delta\boldsymbol{\Omega}\big\}\mathbf{R}-\{\dot{\mathbf{W}}\mathbf{W}-\mathbf{W}\dot{\mathbf{W}}\},\quad \mathbf{Y}\in\mathcal{A}. \tag{17}$$

4.2. Variational formulation

As the above problem is actually solved in a weak sense, let us define the spaces of the virtual velocities:

$$\mathcal{V}(T)=\left\{\vec{\varphi}\in L^2(0,T;\hat{\mathcal{V}}),\ \frac{\mathrm{d}\vec{\varphi}}{\mathrm{d}t}\in L^2\big(0,T;L^2(\mathcal{D}_a)\big)\right\}, \tag{18}$$

with

$$\hat{\mathcal{V}}=\left\{\vec{\varphi}\in H^3(\mathcal{D}_a),\ \vec{\varphi}=0,\ \frac{\partial}{\partial N}(\operatorname{grad}\vec{\varphi})=0,\ \text{on } \Gamma_0,\ \operatorname{grad}\vec{\varphi}=0,\ \text{on } \partial\mathcal{D}_a\right\}. \tag{19}$$

We also introduce the space of the virtual angular velocities:

$$\mathcal{V}_{rv}(T)=\big\{\hat{\boldsymbol{\Omega}}\in L^2\big(0,T;H^1(\mathcal{D}_a)\big),\ \hat{\boldsymbol{\Omega}}\in\mathcal{A},\ \hat{\boldsymbol{\Omega}}=0,\ \text{on } \partial\mathcal{D}_a\big\}. \tag{20}$$

Problem (P). We look for the pair $(\Phi,\mathbf{A})$ fulfilling:

$$\Phi\in L^\infty\big(0,T;H^3(\mathcal{D}_a)\big)\cap H^1\big(0,T;H^1(\mathcal{D}_a)\big)\cap W^{1,\infty}\big(0,T;L^2(\mathcal{D}_a)\big), \tag{21}$$

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2}\in\mathcal{V}'(T),\qquad \mathbf{A}\in\mathcal{V}'_{rv}(T), \tag{22}$$

$$(\Phi-\mathbf{a})\in\mathcal{V}(T), \tag{23}$$

$$\begin{aligned}\forall\vec{\varphi}\in\mathcal{V}(T),\quad &\left\langle\!\!\left\langle\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2},\vec{\varphi}\right\rangle\!\!\right\rangle+\int_0^T\int_{\mathcal{D}_a}\mathbf{R}\left\{(\mathbf{W}-\mathbf{I})+\dot{\mathbf{W}}+\frac{\partial\hat{\Psi}}{\partial\mathbf{W}}(\mathbf{W})\right\}:\operatorname{grad}\vec{\varphi}\,\mathrm{d}a\,\mathrm{d}\tau\\ &+\int_0^T\frac{1}{2}\big\langle\mathbf{A}:\mathbf{R}^T\operatorname{grad}\vec{\varphi}-(\operatorname{grad}\vec{\varphi})^T\mathbf{R}\big\rangle\mathrm{d}\tau\\ &+\int_0^T\int_{\mathcal{D}_a}\operatorname{grad}\Delta\Phi:\operatorname{grad}\Delta\vec{\varphi}\,\mathrm{d}a\,\mathrm{d}\tau=\int_0^T\int_{\mathcal{D}_a}\mathbf{f}\cdot\vec{\varphi}\,\mathrm{d}a\,\mathrm{d}\tau,\end{aligned} \tag{24}$$

$$\boldsymbol{\Omega}=\dot{\mathbf{R}}\mathbf{R}^T\in\mathcal{V}_{rv}(T),\qquad \mathbf{W}=\sqrt{\mathbf{F}^T\mathbf{F}},\qquad \mathbf{F}=\operatorname{grad}\Phi,\qquad \mathbf{RW}=\mathbf{F},\qquad \mathbf{R}(\mathbf{a},0)=\mathbf{I},\quad \mathbf{a}\in\mathcal{D}_a, \tag{25}$$

$$\begin{aligned}\forall\hat{\boldsymbol{\Omega}}\in\mathcal{V}_{rv}(T),\quad &\int_0^T\int_{\mathcal{D}_a}\big\{(\operatorname{grad}\mathbf{R})\mathbf{R}^T:\operatorname{grad}\hat{\boldsymbol{\Omega}}+\operatorname{grad}\boldsymbol{\Omega}:\operatorname{grad}\hat{\boldsymbol{\Omega}}\big\}\,\mathrm{d}a\,\mathrm{d}\tau\\ &=\int_0^T\big\langle\mathbf{A}:\big\{\mathbf{R}^T\hat{\boldsymbol{\Omega}}\mathbf{RW}+\mathbf{WR}^T\hat{\boldsymbol{\Omega}}\mathbf{R}\big\}\big\rangle\mathrm{d}\tau+\int_0^T\int_{\mathcal{D}_a}\mathbf{R}\{\dot{\mathbf{W}}\mathbf{W}-\mathbf{W}\dot{\mathbf{W}}\}\mathbf{R}^T:\hat{\boldsymbol{\Omega}}\,\mathrm{d}a\,\mathrm{d}\tau,\end{aligned} \tag{26}$$

where $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ is the duality pairing between the dual space $\mathcal{V}'(T)$ and $\mathcal{V}(T)$, and $\langle\cdot:\cdot\rangle$ the duality pairing between $H^{-1}(\mathcal{D}_a)$ and $H_0^1(\mathcal{D}_a)$.

4.2.1. The a priori estimates

By use of a suitable Galerkin approximation combined with the principle of virtual power with the actual velocities, we get a priori estimates on Φ, $\mathbf{W}$, $\mathbf{R}$ and $\hat{\Psi}(\mathbf{W})$ but not on reaction matrix $\mathbf{A}$, because it is a reaction to a workless constraint (a *vincolo perfetto* in Italian, a *liaison parfaite* in French). The a priori estimate on $\mathbf{A}$ results from the angular momentum equation of motion equivalent to relationship (16).

4.2.2. The existence theorem

By passing to the limit in the approximating problem, we can prove the following local-in-time existence result. The local-in-time validity of the result depends on the fact that we can control $\partial\hat{\Psi}$ just as long as $\mathbf{W}$ stays away from the boundary of C_α.

Theorem 1. *Assuming that* $\mathbf{f}$ *belongs to* $L^\infty(0,T;L^2(\mathcal{D}_a))$*, there exists some* $\hat{T}$*, with* $0<\hat{T}\leqslant T$*, such that the Problem (P) admits a solution in* $(0,\hat{T})$*. Moreover, the estimates*:

$$\|\Phi\|_{L^\infty(0,\hat{T};H^3(\mathcal{D}_a))\cap H^1(0,\hat{T};H^1(\mathcal{D}_a))\cap W^{1,\infty}(0,\hat{T};L^2(\mathcal{D}_a))}\leqslant c,\qquad \left\|\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2}\right\|_{\mathcal{V}'(\hat{T})}\leqslant c,$$
$$\|\mathbf{W}\|_{L^\infty(0,\hat{T};H^2(\mathcal{D}_a))\cap H^1(0,\hat{T};L^2(\mathcal{D}_a))}\leqslant c,\qquad \|\hat{\Psi}(\mathbf{W})\|_{L^\infty(0,\hat{T};L^1(\mathcal{D}_a))}\leqslant c,$$
$$\|\mathbf{R}\|_{L^\infty(0,\hat{T};H^2(\mathcal{D}_a))\cap H^1(0,\hat{T};H^1(\mathcal{D}_a))}\leqslant c,\qquad \|\mathbf{A}\|_{\mathcal{V}'_{rv}(\hat{T})}\leqslant c,$$

hold for some positive constant c depending on T and on the data of the problem.

5. A regularity result

Due to the regularities we have obtained, we can prove that $\|\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2}\|_{L^2(0,\hat{T};\hat{\mathcal{V}}')}\leqslant c$, where $\hat{\mathcal{V}}'$ is the dual space of $\hat{\mathcal{V}}$ defined in (19). Then, the variational equality (24) can be rewritten in $(0,\hat{T})$ as:

$$\begin{aligned}
&{}_{\hat{\mathcal{V}}'}\left\langle\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2}(t),\vec{\varphi}\right\rangle_{\hat{\mathcal{V}}}+\int_{\mathcal{D}_a}\mathbf{R}\left\{(\mathbf{W}-\mathbf{I})+\dot{\mathbf{W}}+\frac{\partial\hat{\Psi}}{\partial\mathbf{W}}(\mathbf{W})\right\}(t):\operatorname{grad}\vec{\varphi}\,\mathrm{d}a\\
&\quad+\frac{1}{2}\left\langle\mathbf{A}(t),\mathbf{R}^T(t)\operatorname{grad}\vec{\varphi}-(\operatorname{grad}\vec{\varphi})^T\mathbf{R}(t)\right\rangle+\int_{\mathcal{D}_a}\operatorname{grad}\Delta\Phi(t):\operatorname{grad}\Delta\vec{\varphi}\,\mathrm{d}a\\
&=\int_{\mathcal{D}_a}\mathbf{f}(t)\cdot\vec{\varphi}\,\mathrm{d}a\quad\text{for all }\vec{\varphi}\in\hat{\mathcal{V}},\ \text{for a.e. }t\in(0,\hat{T}).
\end{aligned}\tag{27}$$

6. Mechanics and the local-in-time solution

It is impossible to have a global-in-time solution because the modeling is not complete: it does not take into account collisions, i.e., discontinuities of the velocity with respect to time. Even if we have eliminated the possibility of the interruption of the smooth motion resulting from collision with an obstacle or from self-collision, internal collisions are possible, i.e., discontinuities of velocity due to flattening or crushing inside the solid (think of pasta being crushed between two fingers). Thus, it is possible that the motion, the smooth motion, is interrupted at some time $\hat{T}>0$. Internal collisions may be investigated within the collision theory [2], which gives us the possibility to study the velocity after the collision and allows the motion to go on.

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