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Partial differential equations

Asymptotic stability of the semilinear wave equation with boundary damping and source term

Stabilité asymptotique des solutions de l'équation des ondes semi-linéaire avec amortissement et terme source dans les conditions aux limites

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A R T I C L E   I N F O

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A B S T R A C T

In this paper, we consider the semilinear wave equation with boundary conditions. This work is devoted to prove the uniform decay rates of the wave equation with boundary, without imposing any restrictive growth near-zero assumption on the damping term.

R É S U M É

Dans cet article, on considère l'équation des ondes semi-linéaire avec conditions aux limites. L'étude consiste à établir la décroissance uniforme des solutions du problème posé sans imposer de restrictions de croissance sur le terme d'amortissement au voisinage de zéro.

1. Introduction

In this paper, we are concerned with the uniform decay rates of solutions of the semilinear wave equation with boundary damping and source term:

$$\begin{cases} u'' - \mu(t)\Delta u + h(u) = 0 \quad \text{in } \Omega \times (0, +\infty), \\ u = 0 \quad \text{on } \Gamma_0 \times (0, +\infty), \\ \mu(t)\dfrac{\partial u}{\partial \nu} + g(u') = |u|^\gamma u \quad \text{on } \Gamma_1 \times (0, +\infty), \\ u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x), \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ ($n \geqslant 1$) with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ of class C^2. Here, $\Gamma_0 \neq \emptyset$, Γ_0 and Γ_1 are closed and disjoint. Let ν be the outward normal to Γ. Δ stands for the Laplacian with respect to the spatial variables, respectively; $'$ denotes the derivative with respect to time t.

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Consider the problem similar to (1.1), more precisely, given by:

$$\begin{cases} u'' - \mu(t)\Delta u + h(u) = 0 \quad \text{in } \Omega \times (0, +\infty), \\ u = 0 \quad \text{on } \Gamma_0 \times (0, +\infty), \\ \mu(t)\dfrac{\partial u}{\partial \nu} + g\big(u'\big) = p(u) \quad \text{on } \Gamma_1 \times (0, +\infty), \\ u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x). \end{cases} \tag{$*$}$$

The problem of proving uniform decay rates for the solutions to the wave equation with boundary conditions has recently attracted a lot of attention and various results are available (see [2–5,10,13,18,20,21,23] and a list of references therein).

When $\mu = 1$, $h = 0$ and $p = 0$ or $p(s)s \leqslant 0$, the problem $(*)$ was widely studied by many authors: under quite strong assumptions on the damping term g, some of the most important papers are those of Chen [7], Haraux [11], Komornik [12], Lasiecka and Tataru [14], Nakao [17], Zuazua [24]. The condition $p(s)s \leqslant 0$ means that p represents an attractive force. When $p(s)s \geqslant 0$, as in the present case, that is when p represents a source term, the situation is more delicate, since the solutions of $(*)$ can blow up in finite time. To describe briefly the results presented in the literature, we start by the work of Levine and Payne [15], who considered $\mu = 1$, $h = g = 0$, $p = |u|^{\rho-2}u$ and proved that when the initial energy is negative, solutions blow up in finite time. Recently, Ha [8] proved a finite time blow-up result with $g = |u'|^{\rho}u'$, $p = |u|^{\gamma}u$ and a positive initial energy. On the other hand, Vitillaro [22] considered the problem $(*)$ with $\mu = 1$, $h = 0$, $g = |u'|^{\rho-2}u'$ and $p = |u|^{\gamma-2}u$. He proved the existence of solutions except energy decay estimate. Cavalcanti et al. [6] studied the problem $(*)$ considering $\mu = 1$, $h = 0$, $g = |u'|^{\rho}u'$ and $p = \beta|u|^{\gamma}u$. They proved the existence of solutions applying semigroup theory. They also analyzed the boundary stabilization of solution without the function h.

For the case when $\mu \neq 1$ and $h \neq 0$, Araruna and Maciel [1] proved the existence of a solution and of exponential decay by using Lyapunov functional with $p = 0$ and g linear. On the other hand, Park and Ha [19] studied the existence of a solution and of exponential and polynomial decay rates by using a multiplier technique here $p = |u|^{\gamma}u$ and g has a polynomial growth near zero.

When the function g does not have polynomial growth near zero, there were very few results. Martinez [16] studied a linear wave equation with a boundary damping term. He proved the explicit decay estimate of the energy even if the damping term g does not have a polynomial growth near zero. In order to obtain the explicit decay estimate, he used the construction of a special weight function and the generalization of a technique of partition of the boundary. Lasiecka and Tataru [14] studied the more general case of a semilinear wave equation damped with a nonlinear velocity feedback acting on Γ_1, under some very weak geometrical conditions on Γ_0 and Γ_1. Without the assumption that g has a polynomial behavior near zero, they proved that the energy decays as fast as the solution of some associated differential equation. More precisely, they generalized the method used to obtain uniform decay estimates when g has a polynomial behavior near zero. Cavalcanti et al. [3] proved explicit decay rates for the wave equation without any restriction on the growth of the g at the origin using an algorithm in [14] and generalizes substantially the ones considered in [16]. The above-mentioned references handled $\mu(t)$ is a positive constant and used semigroup theory. On the other hand, when $\mu(t)$ is a nonconstant function in time, the techniques developed in the prior literature based on semigroup theory are no longer applicable. In this case, the problem $(*)$ brings new contributions regarding the existence (cf. [19]) as well as uniform decay rate estimates. In this case, the Faedo–Galerkin method is crucial in the well-posedness and the method developed in [16] succeeds in obtaining general decay rate estimates. The method given in [3] is no longer valid, since one has a nonautonomous system.

In this paper, we prove uniform decay rates of solutions to the semilinear wave equation with boundary condition without imposing any restrictive growth near-zero assumption on the damping term. The goal of this paper is to generalize the result of [19] under a weakened assumption, that the damping term g does not necessarily have polynomial growth near zero by applying the method developed in [16].

This paper is organized as follows: In Section 2, we recall the hypotheses to prove our main result and introduce a main result. In Section 3, by using the multiplier technique and Martinez's method [16], we prove the energy decay rates of (1.1).

2. Hypotheses and main result

We begin this section by introducing some hypotheses and our main result. Throughout this paper, we use standard functional spaces and denote that $\|\cdot\|_p$, $\|\cdot\|_{p,\Gamma_1}$ are $L^p(\Omega)$ norm and $L^p(\Gamma_1)$ norm, respectively.

$(\mathbf{H_1})$ Hypotheses on $\boldsymbol{\Omega}$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $n \geqslant 1$, with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ of class C^2. Here Γ_0 and Γ_1 are closed and disjoint, $\Gamma_0 \neq \emptyset$, satisfying the following conditions:

$$\begin{aligned} &m(x)\cdot\nu(x) \geqslant \sigma > 0 \quad \text{on } \Gamma_1, \qquad m(x)\cdot\nu(x) \leqslant 0 \quad \text{on } \Gamma_0, \\ &m(x) = x - x^0 \quad \big(x^0 \in \mathbb{R}^n\big) \quad \text{and} \quad R = \max_{x\in\overline{\Omega}}\big|m(x)\big|, \end{aligned} \tag{2.1}$$

where ν represents the unit outward normal vector to Γ. And we assume that:

$$\mu(0)\frac{\partial u_0}{\partial \nu} + g(u_1) = |u_0|^\gamma u_0 \quad \text{on } \Gamma_1. \tag{2.2}$$

$(\mathbf{H_2})$ Hypotheses on μ, h.

Let $\mu \in W^{1,\infty}(0,+\infty) \cap W^{1,1}(0,+\infty)$ satisfy the following conditions:

$$\mu(t) \geqslant \mu_0 > 0, \qquad \mu'(t) \leqslant 0 \quad \text{a.e. in } [0,+\infty), \tag{2.3}$$

where μ_0 and μ_1 are positive constants. Moreover, we assume that:

$$h : \mathbb{R} \to \mathbb{R} \quad \text{is a Lipschitz function} \quad \text{and} \quad h(s)s \geqslant 0 \quad \text{for all } s \in \mathbb{R} \tag{2.4}$$

and there exists a positive constant δ such that:

$$(n-1)h(s)s \geqslant (2n+\delta)H(s), \tag{2.5}$$

where $H(s) = \int_0^s h(\tau)\,d\tau$.

$(\mathbf{H_3})$ Hypotheses on g.

Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing C^1 function such that $g(0) = 0$ and suppose that there exists a strictly increasing and odd function β of C^1 class on $[-1,1]$ such that:

$$\big|\beta(s)\big| \leqslant \big|g(s)\big| \leqslant \big|\beta^{-1}(s)\big| \quad \text{if } |s| \leqslant 1, \tag{2.6}$$

$$C_1|s| \leqslant \big|g(s)\big| \leqslant C_2|s| \quad \text{if } |s| > 1, \tag{2.7}$$

where β^{-1} denotes the inverse function of β and C_1, C_2 are positive constants. Furthermore,

$$0 \leqslant \gamma < \frac{1}{n-2} \quad \text{if } n \geqslant 3 \quad \text{and} \quad \gamma \geqslant 0 \quad \text{if } n = 1, 2. \tag{2.8}$$

First of all, we consider a well-posedness result. In order to formulate our results, it is convenient to introduce the energy of the system:

$$E(t) = \frac{1}{2}\|u'\|_2^2 + \int_\Omega H\big(u(x,t)\big)\,dx + \frac{1}{2}\mu(t)\|\nabla u\|_2^2 - \frac{1}{\gamma+2}\|u\|_{\gamma+2,\Gamma_1}^{\gamma+2}, \tag{2.9}$$

then

$$E'(t) = \mu'(t)\|\nabla u\|_2^2 - \int_{\Gamma_1} g(u')u'\,d\Gamma.$$

So the energy $E(t)$ is a nonincreasing function.

Our goal is to determine a set of initial data that is invariant with respect to the flow. To achieve this, we introduce the following notations using the potential well theory (see [19]):

- $0 < K_0 := \sup_{u \in H^1_{\Gamma_0}(\Omega),\, u \neq 0}\big(\frac{\|u\|_{\gamma+2,\Gamma_1}}{\|\nabla u\|_2}\big) < +\infty$;
- λ_0 is the first positive zero of the function $j'(\lambda)$, where $j(\lambda) = \frac{\mu_0}{2}\lambda^2 - \frac{1}{\gamma+2}K_0^{\gamma+2}\lambda^{\gamma+2}$; then $\lambda_0 = (\frac{\mu_0}{K_0^{\gamma+2}})^{\frac{1}{\gamma}}$ is the absolute maximum point of j;
- $d := j(\lambda_0)$.

Considering the above notations, we have the following result. Assume that hypotheses (H_1)–(H_3) hold and:

$$(u_0, u_1) \in \big\{(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega);\ \|\nabla u_0\|_2 < \lambda_0,\ E(0) < d\big\}. \tag{2.10}$$

Then the problem (1.1) possesses a unique weak solution in the class:

$$u \in C^0\big(0,T;H^1_{\Gamma_0}(\Omega)\big) \cap C^1\big(0,T;L^2(\Omega)\big).$$

We note that if h is continuous function such that $h(s)s \geqslant 0$, for all $s \in \mathbb{R}$, then the problem (1.1) has at least a weak solution having the same regularity (see [19]). The above existence result can be extended in this paper, even though Park and Ha [19] considered the pure-power damping term. Indeed, [9] proved the same regularity as [19] having the same damping term $g(u')$.

Now we are in a position allowing us to state our result.

Theorem 2.1. *Assume that hypotheses* (H_1), (H_2), (H_3) *and* (2.10) *hold. Then we have*:

$$E(t) \leqslant C_3\left(G^{-1}\left(\frac{1}{t}\right)\right)^2, \tag{2.11}$$

where $G(s) := s\beta(s)$ *and* C_3 *is a positive constant that depends only on* $E(0)$. *Moreover, if the function* $N(s) := \frac{\beta(s)}{s}$ *is nondecreasing on* $[0, \eta]$ *for some* $\eta > 0$ *and* $N(0) = 0$, *then*:

$$E(t) \leqslant C_4\left(\beta^{-1}\left(\frac{1}{t}\right)\right)^2, \tag{2.12}$$

where C_4 *is a positive constant that depends only on* $E(0)$.

To end this section, we recall a technical lemma that will play an essential role when establishing the asymptotic behavior.

Lemma 2.1. *(See [16].) Let* $E : \mathbb{R}_+ \to \mathbb{R}_+$ *be a nonincreasing function and* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *a strictly increasing function of class* C^1 *such that*:

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \to +\infty \quad \text{as } t \to +\infty.$$

Assume that there exist $\sigma > 0$, $\sigma' \geqslant 0$ *and* $C > 0$ *such that*:

$$\int_S^{+\infty} E^{1+\sigma}(t)\phi'(t)\,\mathrm{d}t \leqslant CE^{1+\sigma}(S) + \frac{C}{(1+\phi(S))^{\sigma'}}E^{\sigma}(0)E(S), \quad 0 \leqslant S < +\infty.$$

Then, there exists $C > 0$ *such that*:

$$E(t) \leqslant E(0)\frac{C}{(1+\phi(t))^{(1+\sigma')/\sigma}}, \quad \text{for all } t > 0.$$

3. Proof of Theorem 2.1

In this section, we prove the uniform decay for strong solutions of (1.1), and by considering the density arguments used in the existence of weak solutions, we also can extend our results to weak solutions. In the following section, the symbol C indicates positive constants, which may be different.

By considering $0 \leqslant S < T < +\infty$ from the definition of energy $E(t)$, we obtain:

$$E(T) - E(S) = \frac{1}{2}\int_S^T \mu'(t)\|\nabla u\|_2^2\,\mathrm{d}t - \int_S^T\int_{\Gamma_1} g(u')u'\,\mathrm{d}\Gamma \leqslant 0.$$

So we conclude that $E(t)$ is a nonincreasing function. Now let us multiply Eq. (1.1) by $E(t)\phi'(t)Mu$, where Mu is given by $Mu = 2(m \cdot \nabla u) + (n-1)u$ and $\phi : \mathbb{R} \to \mathbb{R}$ is a concave nondecreasing function of class C^2, such that $\phi(t) \to +\infty$ as $t \to +\infty$, and then integrate the obtained result over $\Omega \times [S, T]$. Then by similar arguments of Section V of [19], we have:

$$\int_S^T E^2\phi'(t)\,\mathrm{d}t \leqslant CE^2(S) + \underbrace{\int_S^T E(t)\phi'(t)\int_{\Gamma_1}(m\cdot\nu)|u'|^2\,\mathrm{d}\Gamma\,\mathrm{d}t}_{:=I_1} + C\underbrace{\int_S^T E(t)\phi'(t)\int_{\Gamma_1} g(u')^2\,\mathrm{d}\Gamma\,\mathrm{d}t}_{:=I_2}. \tag{3.1}$$

Moreover, the last two terms of the right-hand side of (3.1) can be estimated as following by the same arguments as in [16,21]:

$$I_1, I_2 \leqslant CE^2(S) + CE(S)\int_S^T \phi'(t)\big(\beta^{-1}(\phi'(t))\big)^2\,\mathrm{d}t. \tag{3.2}$$

By replacing (3.2) in (3.1), we obtain:

$$\begin{aligned}
\int_S^T E^2(t)\phi'(t)\,\mathrm{d}t &\leqslant CE^2(S) + CE(S)\int_S^T \phi'(t)\big(\beta^{-1}\big(\phi'(t)\big)\big)^2\,\mathrm{d}t \\
&\leqslant CE^2(S) + CE(S)\int_S^{+\infty} \phi'(t)\big(\beta^{-1}\big(\phi'(t)\big)\big)^2\,\mathrm{d}t \\
&\leqslant CE^2(S) + CE(S)\int_{\phi(S)}^{+\infty} \left(\beta^{-1}\left(\frac{1}{\psi'(s)}\right)\right)^2\,\mathrm{d}s \\
&\leqslant CE^2(S) + \frac{C}{\phi(S)}E(S),
\end{aligned}$$

and applying Lemma 2.1 with $\sigma = \sigma' = 1$, we deduce:

$$E(t) \leqslant \frac{C}{\phi^2(t)} \quad \text{for all } t > 0. \tag{3.3}$$

Let s_0 be a number such that $\beta(\frac{1}{s_0}) \leqslant 1$. Since β is nondecreasing, we have:

$$\psi(s) \leqslant 1 + (s-1)\frac{1}{\beta(\frac{1}{s})} \leqslant \frac{1}{G(\frac{1}{s})}, \quad \text{for all } s \geqslant s_0.$$

Consequently, having in mind that $\phi = \psi^{-1}$, the last inequality yields:

$$s \leqslant \phi\left(\frac{1}{G(\frac{1}{s})}\right) = \phi(t) \quad \text{with } t = \frac{1}{G(\frac{1}{s})}.$$

Then:

$$\frac{1}{\phi(t)} \leqslant G^{-1}\left(\frac{1}{t}\right). \tag{3.4}$$

Combining (3.3) and (3.4) we can prove (2.11). Similarly, we obtain (2.12) (for more details, see [16,21]). Thus, the proof of Theorem 2.1 is completed.

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