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The Ostrovsky–Vakhnenko equation: A Riemann–Hilbert approach

L'équation d'Ostrovsky–Vakhnenko : Une approche de type Riemann–Hilbert

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ABSTRACT

We present an inverse scattering transform approach for the (differentiated) Ostrovsky–Vakhnenko equation:

$$u_{txx} - 3u_x + 3u_x u_{xx} + u u_{xxx} = 0.$$

This equation can also be viewed as the short-wave model for the Degasperis–Procesi equation. The approach is based on an associated Riemann–Hilbert problem, which allows us to give a representation for the classical (smooth) solution of the Cauchy problem, to get the principal term of its long-time asymptotics, and also to find, in a natural way, loop soliton solutions.

RÉSUMÉ

Nous présentons une étude par diffusion inverse de l'équation (différentiée) d'Ostrovsky–Vakhnenko :

$$u_{txx} - 3u_x + 3u_x u_{xx} + u u_{xxx} = 0.$$

Cette équation peut aussi se voir comme le modèle « ondes courtes » de l'équation de Degasperis–Procesi. Notre approche consiste à se ramener à l'étude d'un problème de Riemann–Hilbert associé. Elle nous permet d'obtenir une représentation de la solution classique (lisse) du problème de Cauchy et de déterminer le terme principal de l'asymptotique à temps grand de cette solution. Elle permet aussi d'obtenir, de façon naturelle, des solutions solitons de type à boucle.

Version française abrégée

L'objet de cette Note est de présenter une approche de type diffusion inverse pour l'équation non linéaire d'Ostrovsky–Vakhnenko sur la droite. Elle consiste à reformuler le problème de Cauchy sous la forme d'un problème de Riemann–Hilbert

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matriciel associé. Nous supposons que la donnée initiale $u_0(x)$ est une fonction suffisamment lisse, qui tend assez rapidement vers 0 à l'infini et qui vérifie $-u_{0xx}(x)+1>0$ pour tout x.

Sous ces hypothèses, nous obtenons une représentation de la solution $u(x,t)$ du problème de Cauchy en termes de solution d'un problème de Riemann–Hilbert associé. Les données d'un tel problème sont définies dans le plan complexe du paramètre spectral z. Elles sont exprimées en termes de la fonction spectrale $r(z)$ associée à la donnée initiale $u_0(x)$.

Cette représentation de $u(x,t)$ permet d'appliquer la méthode du « col non linéaire » de Deift et Zhou [11] pour étudier son comportement asymptotique pour de grandes valeurs du temps t. Cette étude révèle que le demi-plan $-\infty<x<+\infty$, $t>0$ se partage en deux secteurs, suivant le signe de x/t, où la solution a un comportement asymptotique de nature différente, soit à décroissance rapide, soit à oscillations modulées lentement décroissantes.

Nous montrons, en outre, comment cette approche permet d'introduire, de façon naturelle, des solutions particulières, de type soliton, de cette équation d'Ostrovsky–Vakhnenko. On les obtient en imposant certaines conditions de résidu dans la formulation du problème de Riemann–Hilbert associé. Les solutions correspondantes sont alors nécessairement des fonctions multi-valuées, de type soliton à boucle (*loop soliton*).

1. Introduction

We consider the partial differential equation:

$$u_{txx}-3\kappa u_x+3u_xu_{xx}+uu_{xxx}=0, \tag{1}$$

where $\kappa>0$ is a parameter and $u\equiv u(x,t)$ is real-valued. This equation stems from the short-wave limit—introducing $x'=x/\varepsilon$, $t'=\varepsilon t$, $u'=u/\varepsilon^2$, where ε is a small parameter—of the Degasperis–Procesi (DP) equation [10]:

$$u_t-u_{txx}+3\kappa u_x+4uu_x=3u_xu_{xx}+uu_{xxx}.$$

For $\kappa=1/3$, Eq. (1) reduces, after the change of variables $(u,t)\mapsto(-u,-t)$, to the (differentiated) Vakhnenko equation [16,19]:

$$(u_t+uu_x)_x+u=0. \tag{2}$$

Eq. (2) arises in the context of the propagation of high-frequency waves in a relaxing medium [18,19]. On the other hand, being written in the form:

$$(u_t+c_0u_x+\alpha uu_x)_x=\gamma u, \tag{3}$$

it is also called the "reduced Ostrovsky equation" [17]: it corresponds, in the case $\beta=0$, to the equation:

$$(u_t+c_0u_x+\alpha uu_x+\beta u_{xxx})_x=\gamma u \tag{4}$$

that was derived by Ostrovsky [15] in the study of weakly nonlinear surface and internal waves in a rotating ocean influenced by Earth rotation. Therefore, we find it more correct to call (2) the "Ostrovsky–Vakhnenko equation" (OV), as it is proposed in [9].

Eq. (4) is also known as the "Rotation-Modified KdV equation" (RMKdV); see, e.g., [7,8]. The term γu (kept in the reduced form (3) of the equation) is responsible for a large-scale dispersion due to the influence of Earth rotation (the Coriolis dispersion). In [12], Hunter noted that Eq. (3) arose as the short-wave limit of the RMKdV equation and, more generally, it is the canonical asymptotic equation for genuinely nonlinear waves that are non-dispersive as their wavelength tends to zero. This justifies the name "Ostrovsky–Hunter equation", which is sometimes used for (2) after the change of variables $(u,t)\mapsto(-u,-t)$.

Eq. (1) is (at least, formally) integrable: it possesses a Lax pair representation:

$$\psi_{xxx}=\lambda(-u_{xx}+\kappa)\psi, \qquad \psi_t=\frac{1}{\lambda}\psi_{xx}-u\psi_x+u_x\psi, \tag{5}$$

where $\psi\equiv\psi(x,t,\lambda)$. In [20], the authors introduced a change of variables $u(x,t)=U(X,T)=W_X(X,T)$, $x=x_0+T+W(X,T)$, $t=X$, which reduced (2) to the so-called "transformed Vakhnenko equation":

$$W_{XXT}+(1+W_T)W_X=0. \tag{6}$$

These variables turned out to be convenient for applying Hirota's method for constructing exact soliton solutions to (2) [20,14,23], which are multi-valued functions having the form of one loop (1-soliton) or of many loops (multi-solitons). Another approach to deriving formulas for multi-loop solutions of (1) was proposed in [13], where these solutions were obtained by taking a scaling limit in Hirota-type formulas for the multi-soliton solution of the DP equation.

In [21,22], some form of the inverse scattering method has been applied to (6), which allowed them to show that the loop solitons can be associated, in a standard way, with the eigenvalues of the X-equation of the Lax pair associated with (6). With this respect, we notice that the new variable X is in fact the original time variable t, and thus the formalism of the inverse scattering method is applied in [21,22] in such a way that the t-equation (in the original variables) is used as spectral problem whereas the x-equation provides the "x-evolution" of the spectral data.

In this Note, we present a Riemann–Hilbert (RH) approach to the OV equation, which is based directly on the Lax pair (5), in the form of a pair of 3×3 matrix ODEs. This approach allows us to handle the initial value problem for (1) in a general setting, consistent with the natural physical sense of the variables: the initial data is a function of x, and we are interested in their evolution in t. Under certain conditions on the initial data, we obtain the long time asymptotics of the solution of the Cauchy problem and also show how loop solitons can be retrieved in the framework of our approach.

2. Riemann–Hilbert formalism

Without loss of generality, in what follows we assume that $\kappa=1$ and we consider the Cauchy problem:

$$u_{txx}-3u_x+3u_xu_{xx}+uu_{xxx}=0,\quad x\in(-\infty,\infty),\ t>0, \tag{7a}$$

$$u(x,0)=u_0(x),\quad x\in(-\infty,\infty), \tag{7b}$$

where $u_0(x)$ is smooth and decays sufficiently fast as $|x|\to\infty$, and $-u_{0xx}(x)+1>0$ for all x—then one can show that $u(x,t)$ exists globally and $-u_{xx}(x,t)+1>0$ for all (x,t).

For studying the Cauchy problem, we propose an inverse scattering formalism, where the solution is represented in terms of the solution of an associated RH problem in the complex plane of the spectral parameter. One of the main advantages of such a representation is the possibility to use it efficiently in studying the long-time behavior of the solution, using the nonlinear steepest descent method [4,1,2].

It is convenient to introduce the inverse scattering formalism through a Lax pair having the form of a system of first-order, matrix-valued linear equations, which provides good control on the behavior of dedicated solutions of this system as functions of the spectral parameter.

A similar approach for the DP equation is developed in [2], but its realization (including the asymptotic analysis) in that case differs substantially from the case of Eq. (7a) presented here, see below.

Proposition 2.1. *Eq.* (7a) *is the compatibility condition of the system of* 3×3 *linear equations*:

$$\Phi_x-q\Lambda(z)\Phi=U\Phi, \tag{8a}$$

$$\Phi_t+\left\{uq\Lambda(z)-\Lambda^{-1}(z)\right\}\Phi=V\Phi, \tag{8b}$$

where $\Phi\equiv\Phi(x,t,z)$ *is* 3×3 *matrix valued*, $q:=(-u_{xx}+1)^{1/3}$, $\Lambda(z)=\operatorname{diag}\{z\omega,z\omega^2,z\}$, $\omega=\mathrm{e}^{2\pi\mathrm{i}/3}$,

$$U=\frac{q_x}{3q}\begin{pmatrix}0&1-\omega^2&1-\omega\\1-\omega&0&1-\omega^2\\1-\omega^2&1-\omega&0\end{pmatrix},\qquad V=-uU+\frac{1}{3z}\left\{3\left(\frac{1}{q}-1\right)I+\left(q^2-\frac{1}{q}\right)\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\right\}.$$

I denotes the 3×3 *identity matrix.*

Vector RH problem. We consider the row vector-valued RH problem

$$\begin{aligned}&\mu_-(y,t,z)=\mu_+(y,t,z)S(y,t,z),\quad z\in\Sigma:=\mathbb{R}\cup\omega\mathbb{R}\cup\omega^2\mathbb{R},\\&\mu(y,t,z)\to\begin{pmatrix}1&1&1\end{pmatrix},\quad z\to\infty,\end{aligned} \tag{9}$$

where $\mu(y,t,z)\equiv(\mu_1(y,t,z)\ \mu_2(y,t,z)\ \mu_3(y,t,z))$ is piecewise analytic w.r.t. Σ, and $\mu_\pm$ denotes its limiting values as z approaches the oriented contour Σ from the $\pm$ side. The input data for this problem—the jump matrix S—is defined in terms of the spectral function $r(z)$, $z\in\mathbb{R}$, and uniquely determined by the initial data $u_0(x)$ in the following way:

$$S(y,t,z)=\mathrm{e}^{y\Lambda(z)+t\Lambda^{-1}(z)}S_0(z)\mathrm{e}^{-y\Lambda(z)-t\Lambda^{-1}(z)},\quad z\in\Sigma, \tag{10}$$

where $S_0(z)$ and $r(z)$ are defined as follows:

(a) for $z\in\mathbb{R}$, $S_0(z)=\begin{pmatrix}1&\bar r(z)&0\\-r(z)&1-|r(z)|^2&0\\0&0&1\end{pmatrix}$;

(b) here $r(z)$ is defined from the relation $\Phi_+^{(1)}(x,z)=\Phi_-^{(1)}(x,z)-r(z)\Phi_-^{(2)}(x,z)$, where the $\Phi_\pm^{(j)}(x,z)$'s are the vector solutions $(\Phi_\pm^{(j)}\equiv(\Phi_{\pm1j}\ \Phi_{\pm2j}\ \Phi_{\pm3j})^T)$ of (8a)—with u replaced by u_0—fixed by the boundary conditions for their components:

$$\begin{cases}\Phi_{\pm11}=(1+\mathrm{o}(1))\mathrm{e}^{\omega zx}, & x\to+\infty,\\ \Phi_{\pm21}=\mathrm{o}(1)\mathrm{e}^{\omega zx}, & x\to\pm\infty,\\ \Phi_{\pm31}=\mathrm{o}(1)\mathrm{e}^{\omega zx}, & x\to\operatorname{sign}(z)\infty,\end{cases}\qquad\begin{cases}\Phi_{-12}=\mathrm{o}(1)\mathrm{e}^{\omega^2 zx}, & x\to+\infty,\\ \Phi_{-22}=(1+\mathrm{o}(1))\mathrm{e}^{\omega^2 zx}, & x\to+\infty,\\ \Phi_{-32}=\mathrm{o}(1)\mathrm{e}^{\omega^2 zx}, & x\to\operatorname{sign}(z)\infty;\end{cases} \tag{11}$$

(c) for $z \in \omega\mathbb{R} \cup \omega^2\mathbb{R}$, $S_0(z)$ is defined using (a) and the relation $S_0(z) = C S_0(\omega z) C^{-1}$ with $C = \begin{pmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{pmatrix}$.

Theorem 2.2. *The solution $u(x,t)$ of the Cauchy problem (7) can be expressed, in parametric form, in terms of the solution $\mu(y,t,z) \equiv (\mu_1(y,t,z) \quad \mu_2(y,t,z) \quad \mu_3(y,t,z))$ of the vector-valued RH problem (9)–(11) as follows: $u(x,t) = \hat{u}(y(x,t),t)$, where:*

$$x(y,t) = y + N(y,t) \equiv y + \lim_{z\to 0}\left(\frac{\mu_3(y,t,z)}{\mu_3(y,t,0)} - 1\right)\frac{1}{z}, \qquad \hat{u}(y,t) = \frac{\partial x(y,t)}{\partial t}. \tag{12}$$

Elements of proof. (i) Assuming that $u(x,t)$ solves (7), define a piecewise (w.r.t. Σ) analytic 3×3 matrix-valued function $\tilde{M}(x,t,z)$ through the solution of the Fredholm integral equations ($j,l = 1,2,3$):

$$\tilde{M}_{jl}(x,t,z) = I_{j,l} + \int_{\infty_{j,l}}^{x} \mathrm{e}^{-\lambda_j(z)\int_x^\xi q(\zeta,t)\,\mathrm{d}\zeta}\big[(U\tilde{M})_{j,l}(\xi,t,z)\big]\mathrm{e}^{\lambda_l(z)\int_x^\xi q(\zeta,t)\,\mathrm{d}\zeta}\,\mathrm{d}\xi,$$

where $\lambda_j(z) = z\omega^j$ and

$$\infty_{j,l} = \begin{cases} +\infty, & \text{if } \operatorname{Re}\lambda_j(z) \geqslant \operatorname{Re}\lambda_l(z), \\ -\infty, & \text{if } \operatorname{Re}\lambda_j(z) < \operatorname{Re}\lambda_l(z). \end{cases} \tag{13}$$

These equations provide good control for large z: $\tilde{M} \to I$ as $z\to\infty$, $z \in \mathbb{C}\setminus\Sigma$. On the other hand, $\Phi := \tilde{M}\mathrm{e}^{Q}$, where $Q(x,t,z) = y(x,t)\Lambda(z) + t\Lambda^{-1}(z)$ with $y := x - \int_x^\infty (q(\xi,t)-1)\,\mathrm{d}\xi$, solves (8). Moreover, the limiting values of $M(y,t,z) := \tilde{M}(x(y,t),t,z)$ as z approaches Σ satisfy the jump relation in (9). Notice that the boundary conditions (11) are consistent with the definition of signs in (13).

(ii) Introduce $\Phi_0(x,t,z) = G^{-1}(x,t)\tilde{M}(x,t,z)\mathrm{e}^{y(x,t)\Lambda(z)}$, where:

$$G(x,t) = \begin{pmatrix} \alpha & \beta & \bar{\beta} \\ \bar{\beta} & \alpha & \beta \\ \beta & \bar{\beta} & \alpha \end{pmatrix} \quad \text{with } \alpha = \frac{1}{3}\left(q + 1 + \frac{1}{q}\right) \text{ and } \beta = \frac{1}{3}\left(q + \omega^2 + \frac{\omega}{q}\right),$$

and notice that Φ_0 solves the differential equation $\Phi_{0x} = (\Lambda(z) + U_0)\Phi_0$ with $U_0 = -\frac{zu_{xx}}{3}\begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1&1&1 \\ 1&1&1 \\ 1&1&1 \end{pmatrix}$. The fact that $U_0(x,t,z) \equiv 0$ at $z=0$ implies that Φ_0 can be efficiently controlled for small z. Namely, in terms of $\tilde{M}$, we have:

$$\tilde{M} = G\left(I + z\left\{-\frac{u_x}{3}\Omega + N\hat{\Lambda}\right\} + z^2\left\{-\frac{u}{3}\tilde{\Omega} - \frac{u_x}{3}N\Omega\hat{\Lambda} + \frac{N^2}{2}\hat{\Lambda}^2\right\} + \mathrm{o}(z^3)\right) \quad \text{as } z \to 0, \tag{14}$$

where $N = x - y(x,t) = \int_x^\infty (q(\xi,t)-1)\,\mathrm{d}\xi$, $\hat{\Lambda} = \operatorname{diag}\{\omega, \omega^2, 1\}$, $\Omega = \begin{pmatrix} \omega & \omega & \omega \\ \omega^2 & \omega^2 & \omega^2 \\ 1 & 1 & 1 \end{pmatrix}$, and $\tilde{\Omega} = \hat{\Lambda}\Omega - \Omega\hat{\Lambda}$.

(iii) The expansion (14) is used to represent the solution $u(x,t)$ of the Cauchy problem in terms of the solution of the Riemann–Hilbert problem evaluated at $z=0$. Indeed, introducing $\mu = (1\ 1\ 1)M$, Eqs. (12) follow in view of (14).

It should be noted that in the case of DP, the representation for the solution of the Cauchy problem is different from (12). It stems from the special matrix structure of the solution of the associated RH problem evaluated at a dedicated (non-zero) point of the complex spectral plane (see [2, Theorem 3.1]).

3. Loop solitons

Particular, closed-form, solutions of Eq. (7a) can be obtained, in the framework of the RH method, assuming that the jump conditions are trivial ($S \equiv I$) and adding to (9) nontrivial residue conditions at certain points $z = z_n$, $n = 1, 2, \ldots$:

$$\operatorname{Res}_{z=z_n}\mu_l(y,t,z) = \mu_j(y,t,z_n)v_n^{j,l}\mathrm{e}^{y(\lambda_j(z)-\lambda_l(z))+t(\lambda_j^{-1}(z)-\lambda_l^{-1}(z))}$$

with some scalar constants $v_n^{j,l}$ (different columns have different poles). In order to have a real-valued solution u, the simplest case (taking into account the symmetries) involves six poles, at $z = \rho\mathrm{e}^{\frac{\mathrm{i}\pi}{6}+\frac{\mathrm{i}\pi m}{3}}$, $m = 0,\ldots,5$, with some $\rho > 0$, where the residue conditions have the form:

$$\operatorname{Res}_{z=-\mathrm{i}\rho}\mu_1(y,t,z) = \mu_2(y,t,\mathrm{i}\rho)\gamma\mathrm{e}^{-\sqrt{3}\rho y - \frac{\sqrt{3}}{\rho}t}$$

with some constant $\gamma \equiv |\gamma|\mathrm{e}^{\mathrm{i}\phi}$ (at the other five points, the associated conditions follow by symmetries). Then the RH problem can be explicitly solved, by linear algebra, which gives the following form of μ:

$$\mu = \left(1 + \frac{\alpha}{z+\mathrm{i}\rho} + \frac{\bar{\alpha}\omega}{z+\rho\mathrm{e}^{\frac{\mathrm{i}\pi}{6}}}, 1 + \frac{\bar{\alpha}}{z-\mathrm{i}\rho} + \frac{\alpha\omega^2}{z+\rho\mathrm{e}^{\frac{-\mathrm{i}\pi}{6}}}, 1 + \frac{\alpha\omega}{z-\rho\mathrm{e}^{\frac{\mathrm{i}\pi}{6}}} + \frac{\bar{\alpha}\omega^2}{z-\rho\mathrm{e}^{\frac{-\mathrm{i}\pi}{6}}}\right),$$

where

$$\alpha = 2\sqrt{3}\rho \frac{e^{i\phi}\hat{e} + \hat{e}^2}{1 - 4\cos(\phi - \frac{\pi}{3})\hat{e} + \hat{e}^2} \quad \text{with } \hat{e} = \hat{e}(y,t) = e^{-\sqrt{3}\rho(y + \frac{t}{\rho^2} + y_0)} \text{ and } y_0 = -\frac{1}{\sqrt{3}\rho}\log\frac{|\gamma|}{2\sqrt{3}\rho}.$$

Taking into account the relation $\partial_y N = (1 - \mu_3(y,t,0))/\mu_3(y,t,0)$, that follows from (14), and requiring u to be bounded forces ϕ to have the value $\phi = \pi/3$, which leads, through (12), to the parametric representation of a one-soliton solution:

$$x(y,t) = y + \frac{2\sqrt{3}}{\rho}\frac{\hat{e}}{1+\hat{e}}, \qquad \hat{u}(y,t) = -\frac{6}{\rho^2}\frac{\hat{e}}{(1+\hat{e})^2}.$$

These formulas show that the soliton solution $\hat{u}(y,t)$ is a smooth function in (y,t), whereas in the original variables (x,t), it is a multivalued, loop-type function. This fact suggests the conjecture that the solution of the Cauchy problem (7) with regular initial condition is always associated with a RH problem without any residue conditions, whereas forcing nontrivial residue conditions leads to non-classical (multivalued) solutions. In contrast, the solitary waves for DP are usual smooth solitons [13].

4. Long time asymptotics

The representation of the solution u to the Cauchy problem (7) in terms of an associated RH problem allows applying the nonlinear steepest descent method of Deift and Zhou [11] to study the long-time asymptotics of u. A key feature of this method consists in a series of deformations of the original RH problem: (i) contour deformations and (ii) approximations of the jump matrix. By (10), for $z \in \mathbb{R}$,

$$S(z) = \begin{pmatrix} 1 & e^{-2it\Theta(\zeta,z)}\bar{r}(z) & 0 \\ -e^{2it\Theta(\zeta,z)}r(z) & 1 - |r(z)|^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\zeta := y/t$ and

$$\Theta(\zeta,z) = -\frac{\sqrt{3}}{2}\left(\zeta z - z^{-1}\right).$$

We have similar expressions for $z \in \omega\mathbb{R}$ and $z \in \omega^2\mathbb{R}$:

$$S(z) = \begin{pmatrix} 1 - |r(\omega^2 z)|^2 & 0 & -e^{-2it\Theta(\zeta,\omega^2 z)}r(\omega^2 z) \\ 0 & 1 & 0 \\ e^{2it\Theta(\zeta,\omega^2 z)}\bar{r}(\omega^2 z) & 0 & 1 \end{pmatrix}, \qquad S(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{2it\Theta(\zeta,\omega z)}\bar{r}(\omega z) \\ 0 & -e^{-2it\Theta(\zeta,\omega z)}r(\omega z) & 1 - |r(\omega z)|^2 \end{pmatrix}.$$

The deformations are dictated by the "signature table", which is the distribution of signs of $\operatorname{Im}\Theta(\zeta,\omega^{-j}z)$ near $z \in \omega^j\mathbb{R}$, $j = 0,1,2$, in the complex z-plane, ζ being fixed. They depend in particular on the possible critical points of the imaginary part of the phase $t\Theta(\zeta,z)$ that exist on the real line only for $\zeta < 0$. The deformations reduce the original RH problem, for a large time t, to model RH problems whose solutions can be explicitly computed [11].

The signature table depends on the value of the parameter $\zeta = y/t$, which leads to the dependence of the asymptotics on the value of $x/t \sim y/t$. Moreover, qualitatively different asymptotics are associated with certain ranges of values of x/t, which correspond to sectors in the half-plane $-\infty < x < \infty$, $t > 0$.

On the one hand, the matrix structure of the jump matrix is similar to that in the case of the DP equation, which implies that the long time analysis shares some common steps with that for DP [2]. On the other hand, the signature table is different, being more similar to the case of the short-wave Camassa–Holm equation [6]. Hence, the decomposition of the half-plane $t > 0$, $-\infty < x < \infty$ into domains with a qualitatively different long-time behavior is different from that in the case of DP.

Theorem 4.1. *Let $u(x,t)$ be the solution of the Cauchy problem (7). Then the large t behavior of u is as follows. Let ε be any small positive number.*

(i) *In the sector $x/t > \varepsilon$, $u(x,t)$ tends to 0 with fast decay. More precisely, $u(x,t) = \mathrm{O}(t^{-n})$ for some $n > 1$ depending on the smoothness and on the rate of decay of the initial data $u_0(x)$.*
(ii) *In the sector $x/t < -\varepsilon$, $u(x,t)$ exhibits decaying—of order $\mathrm{O}(t^{-1/2})$—modulated oscillations with coefficients that are functions of x/t given in terms of the associated spectral function $r(z)$:*

$$u(x,t) = \frac{c_1(\varkappa)}{\sqrt{t}}\cos\bigl(c_2(\varkappa)t + c_3(\varkappa)\log t + c_4(\varkappa)\bigr) + \mathrm{O}\bigl(t^{-\alpha}\bigr), \tag{15}$$

for some $\alpha > 1/2$ and with

$$c_1(\varkappa) = -2^{\frac{3}{2}} 3^{\frac{1}{4}} \sqrt{\frac{h(\varkappa)}{\varkappa^3}} \sin\left(\frac{\arg r(\varkappa) - \arg r(-\varkappa)}{2} - \frac{2\pi}{3}\right),$$
$$c_2(\varkappa) = \frac{2\sqrt{3}}{\varkappa},$$
$$c_3(\varkappa) = h(\varkappa),$$
$$c_4(\varkappa) = h \log\frac{8\sqrt{3}}{\varkappa} + \frac{\arg r(\varkappa) + \arg r(-\varkappa)}{2} + \arg\Gamma(-\mathrm{i}h) + \frac{\pi}{4} + \frac{3\varkappa}{\pi}\int_{\varkappa}^{\infty} \frac{\log(1-|r(s)|^2)}{s^2}\,\mathrm{d}s$$
$$+ \frac{1}{\pi}\left(\int_{-\infty}^{-\varkappa} + \int_{\varkappa}^{\infty}\right) \log|\varkappa - s|\,\mathrm{d}\log\bigl(1-\bigl|r(s)\bigr|^2\bigr) + \frac{1}{2\pi}\left(\int_{-\infty}^{-\varkappa} + \int_{\varkappa}^{\infty}\right)\frac{\log(1-|r(s)|^2)(2s+\varkappa)}{s^2 + s\varkappa + \varkappa^2}\,\mathrm{d}s,$$

with $h = h(\varkappa) = -\frac{1}{2\pi}\log(1-|r(\varkappa)|^2)$ *and* $\varkappa = \sqrt{t/|x|}$.

Here Γ is the Euler gamma function. Moreover, the error terms $\mathrm{O}(t^{-n})$ and $\mathrm{O}(t^{-\alpha})$ are uniform in the sectors $x/t > \varepsilon$ and $x/t < -\varepsilon$, respectively.

Matching of the asymptotics. Matching of the asymptotics for positive and negative values of x is provided by the fast decay of the amplitude $c_1(\varkappa)$ in the sector $x/t < -\varepsilon$ as $\varkappa = \sqrt{t/|x|} \to \infty$. Indeed, in this limit, the critical point $\varkappa = \sqrt{t/|x|}$ on the contour of the original RH problem, i.e., on the real line, is growing. Thus, the factor $h(\varkappa)$ in $c_1(\varkappa)$ is decaying to 0 as fast as the reflection coefficient $r(\varkappa)$ is, the latter depending on the smoothness and decay of the initial condition $u_0(x)$. Here, one can see a certain analogy with matching the asymptotics for, e.g., the modified Korteweg–de Vries equation in domains where x/t is approaching $-\infty$: there, a similar behavior of the critical points takes place, see [11].

Such a matching is completely different from that in the case of DP. Indeed, in the latter case, a transition zone exists between the solitonic sector (which, in the case of absence of solitons, becomes the sector of fast decay) and the sector of modulated oscillations; in this zone, where $x/t - 3$ is small, the main asymptotic term is expressed in terms of a solution of the Painlevé II equation.

The appearance of a Painlevé zone in the asymptotics of integrable nonlinear equations is indeed characterized by two factors: (i) at the corresponding point ($x/t = 3$ for DP), there is a bifurcation in the signature table for the associated RH problem giving rise to a self-intersection point on the contour, with a specific behavior of the exponential factors near this point; (ii) the value of the reflection coefficient at the corresponding point κ is non-zero and thus one can define a nontrivial solution of the Painlevé II equation (w.r.t. s) having the asymptotics $r(\kappa)\,\mathrm{Ai}(s)$ as $s \to +\infty$, where $\mathrm{Ai}(s)$ is the Airy function.

Details in the case of the Camassa–Holm equation—which is similar, in this respect, to DP—can be found in [5]. In the case of OV, none of these properties is satisfied for small x/t, i.e., for the zone between the sectors of fast decay and of slow decaying modulated oscillations.

An expanded version with proofs can be found in the arXiv preprint [3].

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