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Harmonic analysis

On estimates for the Fourier transform in the space $L^2(\mathbb{R}^n)$

Radouan Daher, Mohamed El Hamma

Department of Mathematics, Faculty of Sciences Aïn Chock, Hassan II University, Casablanca, Morocco

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ABSTRACT

We obtain new inequalities for the Fourier transform in the space $L^2(\mathbb{R}^n)$, using a generalized spherical mean operator for proving two estimates in certain classes of functions characterized by a generalized continuity modulus.

1. Introduction and preliminaries

This work is based mainly on Titchmarsh's theorem ([8], Theorem 84) in the one-dimensional case. In [2], Abilov et al. proved two useful estimates for the Fourier transform in the space of square integral multivariable functions on certain classes of functions characterized by the generalized continuity modulus, and these estimates are proved by Abilov's for only two variables, using a translation operator.

In this paper, we prove the analog of Abilov's results [2] in the Fourier transform for multivariable functions on $\mathbb{R}^n$. For this purpose, we use a spherical mean operator in the place of the translation operator.

Assume that $L^2(\mathbb{R}^n)$ the space of integrable functions f with the norm:

$$\|f\|_2 = \left(\int_{\mathbb{R}^n} |f(x)|^2 \, dx \right)^{1/2}.$$

The Fourier transform for the function $f \in L^1(\mathbb{R}^n)$ is defined by:

$$\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix.\xi} \, dx.$$

The inverse Fourier transform is defined by the formula:

$$f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{ix.\xi} \, d\xi.$$

The Plancherel theorem provides an extension of the Fourier transform to $L^2(\mathbb{R}^n)$, i.e.:

$$\int_{\mathbb{R}^n} |f(x)|^2 \, dx = \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \, d\xi.$$

Let $j_p(z)$ be a normalized Bessel function of the first kind, i.e.:

E-mail address: elmohamed77@yahoo.fr (M. El Hamma).

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$$j_p(z) = \frac{2^p \Gamma(p+1)}{z^p} J_p(z), \quad \forall z \in \mathbb{R}^+, \tag{1}$$

where $J_p(z)$ is a Bessel function of the first kind.

Consider in $L^2(\mathbb{R}^n)$ the spherical mean operator (see [4]):

$$\mathrm{M}_h f(x) = \frac{1}{w_{n-1}} \int_{\mathrm{S}^{n-1}} f(x+hw)\,\mathrm{d}w,$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$, w_{n-1} its total surface measure with respect to the usual induced measure $\mathrm{d}w$.

The finite differences of the first and higher orders are defined by:

$$\Delta_h f(x) = \mathrm{M}_h f(x) - f(x) = (\mathrm{M}_h - \mathrm{I}) f(x),$$

$$\Delta_h^k f(x) = \Delta_h\big(\Delta_h^{k-1} f(x)\big) = (\mathrm{M}_h - \mathrm{I})^k f(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{i}{k} \mathrm{M}_h^i f(x),$$

where $\mathrm{M}_h^0 f(x) = f(x)$, $\mathrm{M}_h^i f(x) = \mathrm{M}_h(\mathrm{M}_h^{i-1} f(x))$ for $i = 1, 2, \dots, k$ and $k = 1, 2, \dots$, I is the identity operator in $L^2(\mathbb{R}^n)$.

The kth order generalized modulus of continuity of function $f \in L^2(\mathbb{R}^n)$ is defined as:

$$\Omega_k(f, \delta) = \big\| \Delta_h^k f(x) \big\|_2, \quad 0 < h \leqslant \delta.$$

Let $\mathrm{W}_{2,\phi}^{r,k}(\mathrm{D})$ denote the class of functions $f \in \mathrm{L}^2(\mathbb{R}^n)$ such that $\mathrm{D}^i f \in \mathrm{L}^2(\mathbb{R}^n)$, $i = 1, 2, \dots, r$ (in the sense of Levi, see [6]) and

$$\Omega_k\big(\mathrm{D}^r f, \delta\big) = O\big(\phi\big(\delta^k\big)\big),$$

where the operator $\mathrm{D} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and $x = (x_1, x_2, \dots, x_n)$ $\mathrm{D}^0 f = f$, $\mathrm{D}^i f = \mathrm{D}(\mathrm{D}^{i-1} f)$, $i = 1, 2, \dots, r$ and $\phi(t)$ is an arbitrary function defined on $[0, \infty)$ i.e.:

$$\mathrm{W}_{2,\phi}^{r,k}(\mathrm{D}) = \big\{ f \in \mathrm{L}^2\big(\mathbb{R}^n\big), \mathrm{D}^i f \in \mathrm{L}^2\big(\mathbb{R}^n\big),\ \text{such that } \Omega_k\big(\mathrm{D}^r f, \delta\big) = O\big(\phi\big(\delta^k\big)\big), i = 1, 2, \dots, r \big\}.$$

According to [4], we have:

$$\mathrm{M}_h f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{f}(\xi) j_{\frac{n-2}{2}}\big(|\xi| h\big) \mathrm{e}^{\mathrm{i}x.\xi}\,\mathrm{d}\xi$$

and

$$f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{f}(\xi) \mathrm{e}^{\mathrm{i}x.\xi}\,\mathrm{d}\xi,$$

i.e.:

$$\mathrm{M}_h f(x) - f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{f}(\xi) \big(j_{\frac{n-2}{2}}\big(|\xi| h\big) - 1 \big) \mathrm{e}^{\mathrm{i}x.\xi}\,\mathrm{d}\xi.$$

By Parseval's identity, we obtain:

$$\big\| \mathrm{M}_h f(x) - f(x) \big\|_2^2 = \int_{\mathbb{R}^n} \big|\widehat{f}(\xi)\big|^2 \big(j_{\frac{n-2}{2}}\big(|\xi| h\big) - 1 \big)^2\,\mathrm{d}\xi.$$

It is easy to show that any function $f \in \mathrm{W}_{2,\phi}^{r,k}(\mathrm{D})$ implies:

$$\big\| \Delta_h^k \mathrm{D}^r f(x) \big\|_2^2 = \int_{\mathbb{R}^n} |\xi|^{2r} \big(1 - j_{\frac{n-2}{2}}\big(|\xi| h\big)\big)^{2k} \big|\widehat{f}(\xi)\big|^2\,\mathrm{d}\xi. \tag{2}$$

From [3], we have:

$$1 - j_{\frac{n-2}{2}}(r) \asymp \min\big(1, r^2\big), \tag{3}$$

where the symbol $\asymp$ means that the left-hand side is bounded above and below by a positive constant times the right-hand side.

2. Main result

Taking into account what was said in Section 1 for some classes of functions characterized by the generalized modulus of continuity, we can prove two estimates for the integral:

$$\int_{|\xi|\geqslant N} \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi.$$

Proposition 2.1. *Let $r \in \mathbb{N}\cup\{0\}$ and $k \in \mathbb{N}$. If $f \in \mathrm{W}^{r,k}_{2,\phi}(\mathrm{D})$ for some fixed ϕ defined on $[0,\infty)$, then*

$$\int_{|\xi|\geqslant N} \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi = O\left(N^{-2r}\phi^2\left(N^{-k}\right)\right)$$

as $N \to \infty$.

Proof. In the terms of $j_p(z)$, we have (see [1]):

$$1 - j_p(z) = O(1), \quad z \geqslant 1, \tag{4}$$

$$1 - j_p(z) = O\left(z^2\right), \quad 0 \leqslant z \leqslant 1, \tag{5}$$

$$\sqrt{hz}\, J_p(hz) = O(1), \quad hz \geqslant 0. \tag{6}$$

Let $f \in \mathrm{W}^{r,m}_{2,\phi}(\mathrm{D})$. By Hölder inequality, we have:

$$\begin{aligned}
&\int_{|\xi|\geqslant N} \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi - \int_{|\xi|\geqslant N} \left|\widehat{f}(\xi)\right|^2 j_{\frac{n-2}{2}}\left(h|\xi|\right) \mathrm{d}\xi = \int_{|\xi|\geqslant N} \left(1 - j_{\frac{n-2}{2}}\left(h|\xi|\right)\right)\left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi \\
&\quad = \int_{|\xi|\geqslant N} \left(1 - j_{\frac{n-2}{2}}\left(h|\xi|\right)\right)\left|\widehat{f}(\xi)\right|^{2-\frac{1}{k}} \left|\widehat{f}(\xi)\right|^{\frac{1}{k}} \mathrm{d}\xi \\
&\quad \leqslant \left(\int_{|\xi|\geqslant N} \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi\right)^{\frac{2k-1}{2k}} \left(\int_{|\xi|\geqslant N} \left(1 - j_{\frac{n-2}{2}}\left(h|\xi|\right)\right)^{2k}\left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi\right)^{\frac{1}{2k}} \\
&\quad \leqslant \left(\int_{|\xi|\geqslant N} \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi\right)^{\frac{2k-1}{2k}} \left(\int_{|\xi|\geqslant N} \frac{1}{|\xi|^{2r}}\left(1 - j_{\frac{n-2}{2}}\left(h|\xi|\right)\right)^{2k}|\xi|^{2r}\left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi\right)^{\frac{1}{2k}} \\
&\quad \leqslant N^{\frac{-r}{k}}\left(\int_{|\xi|\geqslant N} \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi\right)^{\frac{2k-1}{2k}} \left(\int_{|\xi|\geqslant N} \left(1 - j_{\frac{n-2}{2}}\left(h|\xi|\right)\right)^{2k}|\xi|^{2r}\left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi\right)^{\frac{1}{2k}}.
\end{aligned}$$

From formula (2), we have the inequality:

$$\int_{|\xi|\geqslant N} \left(1 - j_{\frac{n-2}{2}}\left(h|\xi|\right)\right)^{2k}|\xi|^{2r}\left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi \leqslant \left\|\Delta_h^k \mathrm{D}^r f(x)\right\|_2^2.$$

Therefore

$$\int_{|\xi|\geqslant N} \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi \leqslant \int_{|\xi|\geqslant N} \left|\widehat{f}(\xi)\right|^2 j_{\frac{n-2}{2}}\left(h|\xi|\right) \mathrm{d}\xi + N^{\frac{-r}{k}}\left(\int_{|\xi|\geqslant N} \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi\right)^{\frac{2k-1}{2k}} \left\|\Delta_h^k \mathrm{D}^r f(x)\right\|_2^{\frac{1}{k}}.$$

Then

$$\int_{|\xi|\geqslant N} \left(1 - j_{\frac{n-2}{2}}\left(h|\xi|\right)\right)\left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi = O\left(N^{\frac{-r}{k}}\left(\int_{|\xi|\geqslant N} \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi\right)^{\frac{2k-1}{2k}} \left\|\Delta_h^k \mathrm{D}^r f(x)\right\|_2^{\frac{1}{k}}\right).$$

Setting $h = \frac{1}{N}$ in the last inequality and from inequality (3), we obtain:

$$\int_{|\xi|\geqslant N} |\widehat{f}(\xi)|^2 \, d\xi = O(N^{\frac{-r}{k}}) \left(\int_{|\xi|\geqslant N} |\widehat{f}(\xi)|^2 \, d\xi \right)^{\frac{2k-1}{2k}} \phi^{\frac{1}{k}}\left(\left(\frac{1}{N}\right)^k\right).$$

Therefore

$$\int_{|\xi|\geqslant N} |\widehat{f}(\xi)|^2 \, d\xi = O\left(N^{-2r}\phi^2\left(N^{-k}\right)\right).$$

This completes the proof of Proposition 2.1. □

Theorem 2.1. *Let $\phi(t) = t^\nu$. Then*

$$\sqrt{\int_{|\xi|\geqslant N} |\widehat{f}(\xi)|^2 \, d\xi} = O\left(N^{-r-k\nu}\right) \Longleftrightarrow f \in \mathrm{W}^{r,k}_{2,\phi}(\mathrm{D})$$

where $r = 0, 1, \ldots; k = 1, 2, \ldots; 0 < \nu < 2$.

Proof. We prove sufficiency by using Proposition 2.1 let $f \in \mathrm{W}^{r,k}_{2,t^\nu}(\mathrm{D})$, then

$$\left(\int_{|\xi|\geqslant N} |\widehat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} = O\left(N^{-r-k\nu}\right).$$

To prove necessity, let:

$$\sqrt{\int_{|\xi|\geqslant N} |\widehat{f}(\xi)|^2 \, d\xi} = O\left(N^{-r-k\nu}\right)$$

i.e.

$$\int_{|\xi|\geqslant N} |\widehat{f}(\xi)|^2 \, d\xi = O\left(N^{-2r-2k\nu}\right).$$

It is easy to show that there exists a function $f \in L^2(\mathbb{R}^n)$ such that $\mathrm{D}^r f \in L^2(\mathbb{R}^n)$ and

$$\mathrm{D}^r f(x) = \frac{(-1)^n}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |\xi|^r \widehat{f}(\xi) e^{ix.\xi} \, d\xi. \tag{7}$$

From formula (7) and Parseval's identity, we have:

$$\left\| \Delta_h^k \mathrm{D}^r f(x) \right\|_2^2 = \int_{\mathbb{R}^n} \left(1 - j_{\frac{n-2}{2}}(h|\xi|)\right)^{2k} |\xi|^{2r} |\widehat{f}(\xi)|^2 \, d\xi.$$

This integral is split into two:

$$\int_{\mathbb{R}^n} = \int_{|\xi|<N} + \int_{|\xi|\geqslant N} = \mathrm{I}_1 + \mathrm{I}_2$$

where $N = [h^{-1}]$. We estimate them separately from (4); we have:

$$\begin{aligned}
\mathrm{I}_2 &= \int_{|\xi|\geqslant N} |\widehat{f}(\xi)|^2 |\xi|^{2r} \left(1 - j_{\frac{n-2}{2}}(h|\xi|)\right)^{2k} d\xi \\
&= O\left(\int_{|\xi|\geqslant N} |\widehat{f}(\xi)|^2 |\xi|^{2r} \, d\xi \right) \\
&= O\left(\sum_{l=0}^{\infty} \int_{N+l\leqslant|\xi|\leqslant N+l+1} |\xi|^{2r} |\widehat{f}(\xi)|^2 \, d\xi \right)
\end{aligned}$$

$$= O\left(\sum_{l=0}^{\infty}(N+l+1)^{2r}\int_{N+l\leqslant|\xi|\leqslant N+l+1}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi\right)$$

$$= O\left(\sum_{l=0}^{\infty}(N+l+1)^{2r}\left[\int_{|\xi|\geqslant N+l}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi-\int_{|\xi|\geqslant N+l+1}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi\right]\right)$$

$$= O\left(\sum_{l=0}^{\infty}(N+l+1)^{2r}\int_{|\xi|\geqslant N+l}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi-\sum_{l=0}^{\infty}(N+l+1)^{2r}\int_{|\xi|\geqslant N+l+1}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi\right)$$

$$= O\left(N^{2r}\int_{|\xi|\geqslant N}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi+\sum_{l=0}^{\infty}\left((N+l+1)^{2r}-(N+l)^{2r}\right)\int_{|\xi|\geqslant N+l}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi\right)$$

$$= O\left(N^{2r}\int_{|\xi|\geqslant N}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi+\sum_{l=1}^{\infty}(N+l)^{2r-1}\int_{|\xi|\geqslant N+l}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi\right)$$

$$= O\left(N^{2r}N^{-2r-2k\nu}\right)+O\left(\sum_{l=1}^{\infty}(N+l)^{2r-1}(N+l)^{-2r-2k\nu}\right)$$

$$= O\left(N^{-2k\nu}\right)+O\left(N^{-2k\nu}\right)=O\left(h^{2k\nu}\right)$$

i.e.:

$$\mathrm{I}_2=O\left(h^{2k\nu}\right).$$

Now, we estimate I_1, by virtue of (5):

$$\mathrm{I}_1=\int_{|\xi|<N}|\xi|^{2r}\left(1-j_{\frac{n-2}{2}}\left(h|\xi|\right)\right)^{2k}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi$$

$$= O\left(h^{4k}\right)\int_{|\xi|<N}|\xi|^{2r}|\xi|^{4k}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi$$

$$= O\left(h^{4k}\right)\int_{|\xi|<N}|\xi|^{2r+4k}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi$$

$$= O\left(h^{4k}\right)\sum_{l=0}^{N-1}\int_{l\leqslant|\xi|<l+1}|\xi|^{2r+4k}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi$$

$$= O\left(h^{4k}\right)\sum_{l=0}^{N-1}(l+1)^{2r+4k}\int_{l\leqslant|\xi|<l+1}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi$$

$$= O\left(h^{4k}\right)\sum_{l=0}^{N-1}(l+1)^{2r+4k}\left[\int_{|\xi|\geqslant l}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi-\int_{|\xi|\geqslant l+1}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi\right]$$

$$= O\left(h^{4k}\right)\left[1+\sum_{l=0}^{N-1}\left((l+1)^{2r+4k}-l^{2r+4k}\right)\int_{|\xi|\geqslant l}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi\right]$$

$$= O\left(h^{4k}\right)\left[1+\sum_{l=1}^{N-1}l^{2r+4k-1}\int_{|\xi|\geqslant l}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi\right]$$

$$= O\left(h^{4k}\right)\left[1+\sum_{l=1}^{N-1}l^{2r+4k-1}l^{-2r-2k\nu}\right]$$

$$= O\left(h^{4k}\right)\left[1+\sum_{l=0}^{N-1} l^{4k-2k\nu-1}\right]$$
$$= O\left(h^{4k}\right) O\left(N^{4k-2k\nu}\right) = O\left(h^{2k\nu}\right)$$

i.e.:

$$\mathrm{I}_1 = O\left(h^{2k\nu}\right).$$

Combining the estimates for I_1 and I_2 gives:

$$\left\|\Delta_h^k \mathrm{D}^r f(x)\right\|_2 = O\left(h^{k\nu}\right).$$

The necessity is proved. □

Theorem 2.1 in the case $r = 0$ and $k = 1$ can be found in the works of Platonov [7] and Gioev [5].

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