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Algebra/Lie algebras

The groups of automorphisms of the Lie algebras of formally analytic vector fields with constant divergence

Le groupe d'automorphismes de l'algèbre de Lie des champs de vecteurs formellement analytiques à divergence constante

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ABSTRACT

Let $S_n = K[[x_1, \dots, x_n]]$ be the algebra of power series over a field K of characteristic zero, $\mathbb{S}_n^c$ be the group of continuous automorphisms of S_n with constant Jacobian, and $\mathfrak{Div}_n^c$ be the Lie algebra of derivations of S_n with constant divergence. We prove that $\mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{Div}_n^c) = \mathrm{Aut}_{\mathrm{Lie},c}(\mathfrak{Div}_n^c) \simeq \mathbb{S}_n^c$.

RÉSUMÉ

Soit $S_n = K[[x_1, \dots, x_n]]$ l'algèbre des séries formelles sur un corps K de caractéristique zéro, $\mathbf{S}_n^c$ le groupe des automorphismes continus de S_n de jacobien constant et $\mathfrak{Div}_n^c$ l'algèbre de Lie des dérivations de S_n à divergence constante. Nous montrons les identités $\mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{Div}_n^c) = \mathrm{Aut}_{\mathrm{Lie},c}(\mathfrak{Div}_n^c) \simeq \mathbf{S}_n^c$.

1. Introduction

In this paper, K is a field of characteristic zero and K^* is its group of units, and the following notation is fixed:

- $P_n := K[x_1, \dots, x_n]$ is a polynomial algebra, $G_n := \mathrm{Aut}_{K\text{-alg}}(P_n)$ is the group of automorphisms of P_n, $S_n := K[[x_1, \dots, x_n]]$ is the algebra of power series over K, $\mathfrak{m} := (x_1, \dots, x_n)$, S_n^* is the group of units of S_n,
- $\mathbb{S}_n := \mathrm{Aut}_{K\text{-alg},c}(S_n)$ is the group of continuous (with respect to the $\mathfrak{m}$-adic topology) automorphisms of S_n and $\mathbb{S}_n^c := \{\sigma \in \mathbb{S}_n \mid \mathcal{J}(\sigma) \in K\}$ where $\mathcal{J}(\sigma)$ is the Jacobian of σ,
- $\partial_1 := \frac{\partial}{\partial x_1}, \dots, \partial_n := \frac{\partial}{\partial x_n}$ are the partial derivatives (K-linear derivations) of S_n,
- $\mathfrak{s}_n := \mathrm{Der}_K(S_n) = \bigoplus_{i=1}^n S_n \partial_i$ is the Lie algebras of K-derivations of S_n where $[\partial, \delta] := \partial\delta - \delta\partial$, and $D_n := \mathrm{Der}_K(P_n) = \bigoplus_{i=1}^n P_n \partial_i$,
- $\mathcal{D}_n := \bigoplus_{i=1}^n K\partial_i$,
- $H_1 := x_1\partial_1, \dots, H_n := x_n\partial_n$,
- for a derivation $\partial = \sum_{i=1}^n a_i \partial_i \in \mathfrak{s}_n$, $\mathrm{div}(\partial) := \sum_{i=1}^n \frac{\partial a_i}{\partial x_i}$ is the *divergence* of ∂,

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- $\mathfrak{div}_n^0 := \{\partial \in D_n \mid \mathrm{div}(\partial) = 0\}$ and $\mathfrak{Div}_n^0 := \{\partial \in \mathfrak{s}_n \mid \mathrm{div}(\partial) = 0\}$ are the Lie algebras of polynomial, respectively, formally analytic vector fields (derivations) with zero divergence,
- $\mathbf{G}_n := \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{div}_n^0)$ and $\widehat{\mathbf{G}}_n := \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{Div}_n^0)$,
- $\mathfrak{div}_n^c := \{\partial \in D_n \mid \mathrm{div}(\partial) \in K\}$ and $\mathfrak{Div}_n^c := \{\partial \in \mathfrak{s}_n \mid \mathrm{div}(\partial) \in K\}$ are the Lie algebras of polynomial, respectively, formally analytic vector fields (derivations) with constant divergence,
- $\mathbf{G}_n^c := \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{div}_n^c)$ and $\widehat{\mathbf{G}}_n^c := \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{Div}_n^c)$.

2. The groups of automorphisms of the Lie algebras $\mathfrak{div}_n^0$ and $\mathfrak{div}_n^c$

Let $\mathrm{Sh}_1 := \{s_\mu \in \mathrm{Aut}_{K\text{-alg}}(K[x]) \mid s_\mu(x) = x + \mu, \mu \in K\}$.

Theorem 2.1. *(See [3,1].)*

$$\mathbf{G}_n \simeq \begin{cases} G_1/\mathrm{Sh}_1 \simeq K^* & \text{if } n = 1, \\ G_n & \text{if } n \geqslant 2. \end{cases}$$

Theorem 2.1 was announced in [3], where a sketch of the proof is given based on a study of certain Lie subalgebras of $\mathfrak{div}_n^0$ of finite codimension.

Theorem 2.2. *(See [1].)* $\mathbf{G}_n^c \simeq G_n$.

3. The groups of automorphisms of the Lie algebras $\mathfrak{Div}_n^0$ and $\mathfrak{Div}_n^c$

Theorem 3.1. *(See [2,3].)* $\widehat{\mathbf{G}}_n \simeq \mathbb{S}_n^c$ *for* $n \geqslant 2$.

The aim of the paper is to prove the following theorem.

Theorem 3.2.

$$\widehat{\mathbf{G}}_n^c \simeq \begin{cases} G_1 & \text{if } n = 1, \\ \mathbb{S}_n^c & \text{if } n \geqslant 2. \end{cases}$$

Proof. For $n = 1$, $\mathfrak{Div}_1^c = K\partial_1 \oplus KH_1 = \mathfrak{div}_1^c$ and so $\widehat{\mathbf{G}}_1^c = \mathbf{G}_1^c = G_1$, by Theorem 2.2. So, let $n \geqslant 2$.

(i) $\mathbb{S}_n^c \subseteq \widehat{\mathbf{G}}_n^c$ via the group monomorphism (Theorem 3.1 and Theorem 5.1):

$$\mathbb{S}_n^c \to \widehat{\mathbf{G}}_n^c, \quad \sigma \mapsto \sigma : \partial \mapsto \sigma(\partial) := \sigma\partial\sigma^{-1}.$$

(ii) $\mathfrak{Div}_n^0 = [\mathfrak{Div}_n^c, \mathfrak{Div}_n^c]$: The equality follows from the fact that $\mathfrak{Div}_n^0$ is a simple Lie algebra which is an ideal of the Lie algebra $\mathfrak{Div}_n^c$ and $\mathfrak{Div}_n^c = \mathfrak{Div}_n^0 \oplus KH_1$.

(iii) The short exact sequence of group homomorphisms:

$$1 \to F := \mathrm{Fix}_{\widehat{\mathbf{G}}_n^c}(\mathfrak{Div}_n^0) \to \widehat{\mathbf{G}}_n^c \overset{\mathrm{res}}{\to} \widehat{\mathbf{G}}_n \to 1$$

is exact (by (i) and Theorem 3.1) where $\mathrm{res} : \sigma \mapsto \sigma|_{\mathfrak{Div}_n^0}$ is the restriction map, see (ii).

(iv) Since $\widehat{\mathbf{G}}_n = \mathbb{S}_n^c$ (Theorem 3.1) and $\mathbb{S}_n^c \subseteq \widehat{\mathbf{G}}_n^c$ (by (i)), the short exact sequence splits:

$$\widehat{\mathbf{G}}_n^c \simeq \widehat{\mathbf{G}}_n \ltimes F. \tag{1}$$

(v) $F = \{e\}$ (Lemma 5.2). Therefore, $\widehat{\mathbf{G}}_n^c \simeq \mathbb{S}_n^c$. □

The Lie algebra $\mathfrak{Div}_n^c$ is a topological Lie algebra with respect to the $\mathfrak{m}$-adic topology, i.e. the set $\{\mathfrak{m}^i\mathfrak{Div}_n^c\}_{i\in\mathbb{N}}$ is a base of open neighbourhoods of zero. Let $\widehat{\mathbf{G}}_{n,top}^c$ be group of automorphisms of the topological Lie algebra $\mathfrak{Div}_n^c$. Clearly, $\widehat{\mathbf{G}}_{n,top}^c \subseteq \widehat{\mathbf{G}}_n^c$. The inverse inclusion follows from Theorem 3.2.

Corollary 3.3. $\widehat{\mathbf{G}}_{n,top}^c = \widehat{\mathbf{G}}_n^c$.

4. The group $\mathbb{S}_n$

Every continuous automorphism $\sigma \in \mathbb{S}_n$ is uniquely determined by the elements:

$$x_1' := \sigma(x_1), \dots, x_n' := \sigma(x_n)$$

that necessarily (by the continuity of σ) belong to the maximal ideal $\mathfrak{m}$ of the algebra S_n, and for all series $f = f(x_1, \dots, x_n) \in S_n$, $\sigma(f) = f(x'_1, \dots, x'_n)$. Let $M_n(S_n)$ be the algebra of $n \times n$ matrices over S_n. The matrix $J(\sigma) := (J(\sigma)_{ij}) \in M_n(S_n)$, where $J(\sigma)_{ij} = \frac{\partial x'_j}{\partial x_i}$, is called the *Jacobian matrix* of σ and its determinant $\mathcal{J}(\sigma) := \det J(\sigma)$ is called the *Jacobian* of σ. So, the jth column of $J(\sigma)$ is the *gradient* $\operatorname{grad} x'_j := (\frac{\partial x'_j}{\partial x_1}, \dots, \frac{\partial x'_j}{\partial x_n})^T$ of the series x'_j. Then the derivations:

$$\partial'_1 := \sigma \partial_1 \sigma^{-1}, \dots, \partial'_n := \sigma \partial_n \sigma^{-1}$$

are the partial derivatives of S_n with respect to the variables $x'_1, \dots, x'_n$,

$$\partial'_1 = \frac{\partial}{\partial x'_1}, \dots, \partial'_n = \frac{\partial}{\partial x'_n}. \tag{2}$$

Every derivation $\partial \in \mathfrak{s}_n$ is a unique sum $\partial = \sum_{i=1}^n a_i \partial_i$ where $a_i = \partial * x_i \in S_n$. Let $\partial := (\partial_1, \dots, \partial_n)^T$ and $\partial' := (\partial'_1, \dots, \partial'_n)^T$ where T stands for the transposition. Then

$$\partial' = J(\sigma)^{-1}\partial, \quad \text{i.e.} \quad \partial'_i = \sum_{j=1}^n \bigl(J(\sigma)^{-1}\bigr)_{ij} \partial_j \quad \text{for } i = 1, \dots, n. \tag{3}$$

In more detail, if $\partial' = A\partial$ where $A = (a_{ij}) \in M_n(S_n)$, i.e. $\partial_i = \sum_{j=1}^n a_{ij}\partial_j$. Then for all $i, j = 1, \dots, n$,

$$\delta_{ij} = \partial'_i * x'_j = \sum_{k=1}^n a_{ik} \frac{\partial x'_j}{\partial x_k}$$

where δ_{ij} is the Kronecker delta function. The equalities above can be written in the matrix form as $AJ(\sigma) = 1$, where 1 is the identity matrix. Therefore, $A = J(\sigma)^{-1}$.

For all $\sigma, \tau \in \mathbb{S}_n$,

$$J(\sigma\tau) = J(\sigma) \cdot \sigma\bigl(J(\tau)\bigr). \tag{4}$$

By taking the determinants of both sides of (4), we have a similar equality of the Jacobians: for all $\sigma, \tau \in \mathbb{S}_n$,

$$\mathcal{J}(\sigma\tau) = \mathcal{J}(\sigma) \cdot \sigma\bigl(\mathcal{J}(\tau)\bigr). \tag{5}$$

By putting $\tau = \sigma^{-1}$ in (4) and (5), we see that $J(\sigma) \in \mathrm{GL}_n(S_n)$, $\mathcal{J}(\sigma) \in S_n^*$, and

$$J\bigl(\sigma^{-1}\bigr) = \sigma^{-1}\bigl(J(\sigma)^{-1}\bigr), \tag{6}$$

$$\mathcal{J}\bigl(\sigma^{-1}\bigr) = \sigma^{-1}\bigl(\mathcal{J}(\sigma)^{-1}\bigr). \tag{7}$$

$$\mathbb{S}_n = \bigl\{\sigma \in \mathrm{End}_{K\text{-alg},c}(S_n) \bigm| \mathcal{J}(\sigma) \in S_n^*\bigr\} = \bigl\{\sigma \in \mathrm{End}_{K\text{-alg},c}(S_n) \bigm| \sigma(x) = Ax + \cdots, A = (a_{ij}) \in \mathrm{GL}_n(K)\bigr\},$$

that is $\sigma(x_i) = \sum_{j=1}^n a_{ij}x_j + \cdots$, where the three dots mean smaller terms ($\dots \in \mathfrak{m}^2$).

Lemma 4.1. *For all* $\sigma \in \mathbb{S}_n^c$,

$$\sum_{j=1}^n \partial_j * \bigl(J(\sigma)^{-1}\bigr)_{ij} = 0 \quad \textit{for } i = 1, \dots, n.$$

Proof. By (3), $\partial'_i = \sum_{i=1}^n (J(\sigma)^{-1})_{ij}\partial_j$. By Theorem 3.1, we have the result. □

5. The divergence commutes with automorphisms $\mathbb{S}_n^c$

The following theorem shows that the divergence commutes with automorphisms $\mathbb{S}_n^c$, i.e. the divergence map $\mathrm{div} : \mathfrak{s}_n \to S_n$ is an $\mathbb{S}_n^c$-module homomorphism.

Theorem 5.1. *For all* $\sigma \in \mathbb{S}_n^c$ *and* $\partial \in \mathfrak{s}_n$,

$$\mathrm{div}\bigl(\sigma(\partial)\bigr) = \sigma\bigl(\mathrm{div}(\partial)\bigr).$$

Proof. Let $\partial = \sum_{i=1}^n a_i \partial_i$ where $a_i \in S_n$. Then $\partial' = \sigma \partial \sigma^{-1} = \sum_{i=1}^n \sigma(a_i)\partial'_i$ where, by (3), $\partial'_i = \sum_{j=1}^n (J(\sigma)^{-1})_{ij}\partial_j$. Now, by Lemma 4.1,

$$\begin{aligned}\operatorname{div}(\partial') &= \sum_{i,j=1}^n \partial_j * \left(\left(J(\sigma)^{-1}\right)_{ij} \sigma(a_i)\right) = \sum_{i=1}^n \left(\sum_{j=1}^n \partial_j * \left(J(\sigma)^{-1}\right)_{ij}\right) \cdot \sigma(a_i) + \sum_{i=1}^n \sum_{j=1}^n \left(J(\sigma)^{-1}\right)_{ij} \partial_j * \sigma(a_i) \\ &= \sum_{i=1}^n \partial'_i * \sigma(a_i) = \sum_{i=1}^n \sigma \partial_i \sigma^{-1} \sigma(a_i) = \sigma\left(\sum_{i=1}^n \partial_i(a_i)\right) = \sigma\left(\operatorname{div}(\partial)\right). \qquad \square\end{aligned}$$

Lemma 5.2. $\mathrm{Fix}_{\widehat{\mathbb{G}}_n^c}(\mathfrak{Div}_n^0) = \{e\}$ *for* $n \geqslant 2$.

Proof. Let $\sigma \in F := \mathrm{Fix}_{\widehat{\mathbb{G}}_n^c}(\mathfrak{Div}_n^0)$, $H'_1 := \sigma(H_1), \dots, H'_n := \sigma(H_n)$. Since $\mathfrak{Div}_n^c = \mathfrak{Div}_n^0 \oplus KH_i$, $i = 1, \dots, n$, it suffices to show that $\sigma(H_i) = H_i$ for $i = 1, \dots, n$. For $i \neq j$, $\sigma(H_i - H_j) = H_i - H_j$, and so $d := H'_i - H_i = H'_j - H_j$. For all $i = 1, \dots, n$,

$$[\partial_i, d] = \sigma\left([\partial_i, H_i]\right) - [\partial_i, H_i] = \sigma(\partial_i) - \partial_i = \partial_i - \partial_i = 0.$$

So, $d \in C_{\mathfrak{Div}_n^c}(\mathcal{D}_n) = \mathcal{D}_n$ (since $C_{\mathfrak{s}_n}(\mathcal{D}_n) = \mathcal{D}_n$) and $d = \sum_{i=1}^n \lambda_i \partial_i$ for some $\lambda_i \in K$ where $C_{\mathcal{G}}(\mathcal{H}) := \{g \in \mathcal{G} \mid [g, \mathcal{H}] = 0\}$ is the centralizer of a subset $\mathcal{H}$ of a Lie algebra $\mathcal{G}$. The elements $H'_1 = H_1 + d, \dots, H'_n = H_n + d$ commute, hence $d = 0$. Therefore, $\sigma = e$. $\square$

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References

[1] V.V. Bavula, The groups of automorphisms of the Lie algebras of polynomial vector fields with zero or constant divergence, arXiv:1310.4741.
[2] A.N. Rudakov, Automorphism groups of infinite-dimensional simple Lie algebras, Izv. Akad. Nauk SSSR, Ser. Mat. 33 (1969) 748–764.
[3] A.N. Rudakov, Subalgebras and automorphisms of Lie algebras of Cartan type, Funkc. Anal. Prilozh. 20 (1) (1986) 83–84.