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Well-posedness and approximation of a measure-valued mass evolution problem with flux boundary conditions

Le caractère bien posé et l'approximation d'un problème d'évolution des mesures de masse avec des conditions frontières sur le flux

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A R T I C L E   I N F O

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A B S T R A C T

This Note deals with imposing a flux boundary condition on a non-conservative measure-valued mass evolution problem posed on a bounded interval. To establish the well-posedness of the problem, we exploit particle system approximations of the mass accumulation in a thin layer near the active boundary. We derive the convergence rate for the approximation procedure as well as the structure of the flux boundary condition in the limit problem.

R É S U M É

Dans cette Note, nous étudions l'évolution de mesures (de masse) dans un intervalle borné où la dynamique non conservative est imposée à l'aide de conditions frontières de type flux. Nous montrons le caractère bien posé du problème en exploitant des systèmes de particules et l'accumulation de masse provoquée par ces particules dans une couche limite tout près de la frontière active. Finalement, nous obtenons la vitesse de convergence de la procedure d'approximation ainsi que la structure de la condition de frontière concernant le problème limite.

1. Introduction

We consider particles moving in the interval $[0,1]$, forced by an externally determined velocity field $v(x)$, $x \in [0,1]$ (see [1,4] for closely related scenarios). There is no interaction among individuals and the boundary $x = 1$ is '*sticking*' and *partially absorbing*: once a particle arrives at the boundary $x = 1$, it stays there and can be removed from the system (being 'absorbed' or 'gated') randomly at a time after arrival that is exponentially distributed with a constant absorption rate $a \geqslant 0$. If a particle is distributed initially according to the probability measure ν_0, then—formally—the distribution of this particle at time t is described by:

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$$\frac{\partial}{\partial t}\mu_t + \frac{\partial}{\partial x}(v\mu_t) = -a\mu_t\big(\{1\}\big)\delta_1, \qquad \mu_0 = \nu_0. \tag{1}$$

Eq. (1) may also be viewed as unifying both a continuum formulation and particle description for this mass evolution problem in a single framework. Note that in the measure-valued formulation for the associated particle system with 'sticking' boundary conditions [8], such Robin-like boundary conditions should be incorporated in the measure-valued equation (1) as a density-dependent (point located) sink.

One may apply the weak solution concept to (1), as in e.g. [3]. In this Note, however, we point out that an approach through *mild solutions* (see e.g. [7]) is feasible, while (1) can be obtained as limit of a family of systems with interaction in a shrinking boundary layer. That is, if $\overline{\mathcal{M}}([0,1])_{\mathrm{BL}}$ is the Banach space completion of the finite Borel measures $\mathcal{M}([0,1])$ on $[0,1]$ for the norm:

$$\|\mu\|^*_{\mathrm{BL}} := \sup\Bigg\{\bigg|\int_{[0,1]} f\,\mathrm{d}\mu\bigg| : \ f\in \mathrm{BL}\big([0,1]\big),\ \|f\|_\infty \leqslant 1,\ |f|_L \leqslant 1\Bigg\},$$

where $\mathrm{BL}([0,1])$ is the space of bounded Lipschitz functions f with Lipschitz constant $|f|_{\mathrm{L}}$ (cf. [6]), then a mild solution to (1) is a continuous map $t\mapsto \mu_t$ from $[0,T]$ into $\overline{\mathcal{M}}([0,1])_{\mathrm{BL}}$ such that:

$$\mu_t = P_t\mu_0 - a\int_0^t \mu_s\big(\{1\}\big)\,\mathrm{d}s\cdot\delta_1. \tag{2}$$

Here, $(P_t)_{t\geqslant 0}$ is the strongly continuous semigroup in $\overline{\mathcal{M}}([0,1])_{\mathrm{BL}}$ associated with mass transport along with characteristics defined by the bounded Lipschitz velocity field $v(x)$. Eq. (2) may be viewed as a usual Variation of Constants formula:

$$\mu_t = P_t\mu_0 + \int_0^t P_{t-s}F(\mu_s)\,\mathrm{d}s, \qquad F(\mu) = -a\mu\big(\{1\}\big)\cdot\delta_1, \tag{3}$$

since δ_1 is invariant under $(P_t)_{t\geqslant 0}$, i.e. $P_t\delta_1 = \delta_1$ for all t. The key point here is that F *is not Lipschitz continuous, not even continuous on* $\overline{\mathcal{M}}([0,1])_{\mathrm{BL}}$, although $t\mapsto\mu_t(\{1\})$ is measurable, such that (2) and (3) are well defined. The standard arguments for solving such equations use Picard iteration, and require Lipschitz continuity of the perturbation term to invoke Banach's Fixed Point Theorem and Gronwall's Lemma; see e.g. [2,3]. Still, problem (2) is well-posed for mild solutions, as will be shown below.

If $x(\cdot,x_0)$ is the unique solution to $\dot{x}(t) = v(x(t))$, $x(0) = x_0$ and $\tau_\partial(x_0)\in[0,\infty]$ is the time at which this solution reaches the boundary 0 or 1, then:

$$\Phi_t(x_0) := \begin{cases} x(t;x_0), & \text{if } t\in I_{x_0},\\ x(\tau_\partial(x_0);x_0), & \text{otherwise,}\end{cases} \tag{4}$$

yields the *stopped individualistic flow* $(\Phi_t)_{t\geqslant 0}$ in $[0,1]$ along characteristics. P_t is the lift of Φ_t to $\mathcal{M}([0,1])$ by means of push forward under Φ_t: for all $\mu\in\mathcal{M}([0,1])$, $P_t\mu := \Phi_t\#\mu = \mu\circ\Phi_t^{-1}$. P_t maps positive measures to positive measures and is mass preserving on positive measures. That is, $(P_t)_{t\geqslant 0}$ is a *Markov semigroup* on measures on $[0,1]$. One has $\|P_t\mu\|_{\mathrm{TV}} \leqslant \|\mu\|_{\mathrm{TV}}$ and $\|P_t\mu\|^*_{\mathrm{BL}} \leqslant e^{|v|_L}\|\mu\|^*_{\mathrm{BL}}$ for general $\mu\in\mathcal{M}([0,1])$.

As a (mild) solution to (1), we consider any continuous map $\mu:\mathbb{R}^+\to\mathcal{M}([0,1])_{\mathrm{BL}}$ that satisfies the integral equation (3), provided $(\Phi_t)_{t\geqslant 0}$ is the stopped individualistic flow defined as in (4) with $P_t\mu := \Phi_t\#\mu$ for all $\mu\in\mathcal{M}([0,1])$.

2. Well-posedness results

Proposition 1 *(Uniqueness). A solution to* (2) *in* $C(\mathbb{R}^+,\mathcal{M}([0,1])_{\mathrm{BL}})$ *is unique, if it exists.*

Proof. A modified argument of Gronwall-type shows the uniqueness of solutions. In fact, if (2) had two solutions μ_t and $\hat{\mu}_t$ on $[0,T]$, having the same initial data μ_0, then for all $t\geqslant 0$,

$$\mu_t - \hat{\mu}_t = -a\int_0^t \big[\mu_s\big(\{1\}\big) - \hat{\mu}_s\big(\{1\}\big)\big]\,\mathrm{d}s\cdot\delta_1. \tag{5}$$

That is, two solutions can differ by mass concentrated at 1 only. Note that the integrand in (5) is a bounded measurable function. Evaluating the latter equation at $\{1\}$ yields: $\mu_t(\{1\}) - \hat{\mu}_t(\{1\}) = -a\int_0^t[\mu_s(\{1\}) - \hat{\mu}_s(\{1\})]\,\mathrm{d}s$, and consequently $|\mu_t(\{1\}) - \hat{\mu}_t(\{1\})| \leqslant a\int_0^t|\mu_s(\{1\}) - \hat{\mu}_s(\{1\})|\,\mathrm{d}s$. A version of Gronwall's Lemma yields that $|\mu_t(\{1\}) - \hat{\mu}_t(\{1\})| = 0$ for all $t\geqslant 0$.

Since there is no smoothing effect in the dynamics in the interior of the interval $[0,1]$, Dirac masses stay Dirac masses. P_t acts simply on Dirac masses: $P_t\delta_x = \delta_{\Phi_t(x)}$. Thus, the solution to (2) with $\mu_0 = \alpha_x(0)\delta_x$ is of the form $\mu_t = \alpha_x(t)\delta_{\Phi_t(x)}$ with $\alpha_x(t) = \alpha_x(0) - a\int_{\tau_\partial(x)\wedge t}^t \alpha_x(s)\,\mathrm{d}s$. Here, $\tau_\partial(x)\wedge t = \min(\tau_\partial(x), t)$ is the amount of time in the interval $[0,t]$ that the individual is not on the boundary. The latter equation is easily solved. Using the linearity of (2) and the identity $\mu = \int_{[0,1]}\delta_x\,\mathrm{d}\mu$ as Bochner integral in $\overline{\mathcal{M}}([0,1])$ (e.g. [6]), one obtains:

Proposition 2 *(Existence). For each $\mu_0 \in \mathcal{M}([0,1])$, there exists a continuous solution $\mu : \mathbb{R}^+ \to \mathcal{M}([0,1])_{\mathrm{BL}}$ to (2) given by the Bochner integral*

$$\mu_t := \int_{[0,1]} \mathrm{e}^{-a[t-\tau_\partial(x)\wedge t]}\delta_{\Phi_t(x)}\,\mathrm{d}\mu_0(x) \quad \text{in } \overline{\mathcal{M}}\big([0,1]\big)_{\mathrm{BL}}. \tag{6}$$

Proof. The integrand in (6) is a bounded continuous function from $[0,1]$ into $\mathcal{M}^+([0,1])_{\mathrm{BL}}$ (cf. Corollary 2.4 in [5]). Thus, for $\mu_0 \in \mathcal{M}^+([0,1])$, the Bochner integral exists, with value in $\mathcal{M}^+([0,1])$, because this cone is closed. For $\mu \in \mathcal{M}([0,1])$, the integral yields a *measure* in $\mathcal{M}([0,1]) \subset \overline{\mathcal{M}}([0,1])_{\mathrm{BL}}$, by using the Jordan decomposition $\mu_0 = \mu_0^+ - \mu_0^-$. Fix $t_0 \in \mathbb{R}^+$ and let $t \in \mathbb{R}^+$. Then:

$$\begin{aligned}
\|\mu_t - \mu_{t_0}\|_{\mathrm{BL}}^* &\leqslant \int_{[0,1]} \big\|\mathrm{e}^{-a[t-\tau_\partial(x)\wedge t]}(\delta_{\Phi_t(x)} - \delta_{\Phi_{t_0}(x)})\big\|_{\mathrm{BL}}^*\,\mathrm{d}|\mu_0|(x) \\
&\quad + \int_{[0,1]} \big|\mathrm{e}^{-a[t-\tau_\partial(x)\wedge t]} - \mathrm{e}^{-a[t_0-\tau_\partial(x)\wedge t_0]}\big|\,\mathrm{d}|\mu_0|(x) \\
&\leqslant \int_{[0,1]} \|\delta_{\Phi_t(x)} - \delta_{\Phi_{t_0}(x)})\|_{\mathrm{BL}}^* + \big|\mathrm{e}^{-a[t-\tau_\partial(x)\wedge t]} - \mathrm{e}^{-a[t_0-\tau_\partial(x)\wedge t_0]}\big|\,\mathrm{d}|\mu_0|(x).
\end{aligned}$$

Continuity of the maps $t \mapsto \Phi_t(x)$ and $t \mapsto \exp(-a[t - \tau_\partial(x)\wedge t])$ and application of Lebesgue's Dominated Convergence Theorem yield continuity of $t \mapsto \mu_t$. We verify easily that μ_t satisfies (2). □

Since the perturbation is not Lipschitz continuous, the standard Gronwall-like argument to obtain continuous dependence on initial conditions fails in this setting. Instead, we use (6). □

Proposition 3 *(Continuous dependence on initial conditions). For each $T \geqslant 0$, there exists $C_T > 0$ such that for all initial measures $\mu_0, \mu_0' \in \mathcal{M}^+([0,1])$, the corresponding solutions μ and μ' to (2) satisfy for all $t \in [0,T]$:*

$$\|\mu_t - \mu_t'\|_{\mathrm{BL}}^* \leqslant C_T\|\mu_0 - \mu_0'\|_{\mathrm{BL}}^*. \tag{7}$$

Proof. In view of Lemma 2.2(*ii*) in [5], we need to control the integral term in (2). It is the total amount of mass that disappeared from the system in the time interval $[0,t]$. To be precise, according to (2) and (6):

$$a\int_0^t \mu_s\big(\{1\}\big)\,\mathrm{d}s = P_t\mu_0(S) - \mu_t(S) = \|\mu_0\|_{\mathrm{TV}} - \mathrm{e}^{-at}\int_{[0,1]} \mathrm{e}^{a[\tau_\partial(x)\wedge t]}\,\mathrm{d}\mu_0(x) = \|\mu_0\|_{\mathrm{BL}}^* - \mathrm{e}^{-at}\big\langle\mu_0, \mathrm{e}^{a[\tau_\partial(\cdot)\wedge t]}\big\rangle. \tag{8}$$

Note that the map $x \mapsto e^{a[\tau_\partial(x)\wedge t]}$ is bounded Lipschitz (Corollary 2.4 in [5]):

$$\big|\mathrm{e}^{a[\tau_\partial(\cdot)\wedge t]}\big|_{\mathrm{L}} \leqslant \mathrm{e}^{at}\big|\tau_\partial(\cdot)\wedge t\big|_{\mathrm{L}} \leqslant e^{at}\big|\tau_\partial(\cdot)\big|_{\mathrm{L}} \quad \text{and} \quad \big\|\mathrm{e}^{a[\tau_\partial(\cdot)\wedge t]}\big\|_\infty \leqslant \mathrm{e}^{at}.$$

Therefore, using Lemma 2.2. from [5] and (8),

$$\begin{aligned}
\|\mu_t - \mu_t'\|_{\mathrm{BL}}^* &\leqslant \big\|P_t\big(\mu_0 - \mu_0'\big)\big\|_{\mathrm{BL}}^* + \left|a\int_0^t \mu_s\big(\{1\}\big) - \mu_s'\big(\{1\}\big)\,\mathrm{d}s\right| \\
&\leqslant \mathrm{e}^{|v|_{\mathrm{L}}t}\big\|\mu_0 - \mu_0'\big\|_{\mathrm{BL}}^* + \big|\|\mu_0\|_{\mathrm{BL}}^* - \big\|\mu_0'\big\|_{\mathrm{BL}}^*\big| + \mathrm{e}^{-at}\big|\big\langle\mu_0 - \mu_0', \mathrm{e}^{a[\tau_\partial(\cdot)\wedge t]}\big\rangle\big| \\
&\leqslant \big\|\mu_0 - \mu_0'\big\|_{\mathrm{BL}}^*\big[\mathrm{e}^{|v|_{\mathrm{L}}t} + 1 + \mathrm{e}^{-at}\big\|\mathrm{e}^{a[\tau_\partial(\cdot)\wedge t]}\big\|_{\mathrm{BL}}\big] \\
&\leqslant \big(\mathrm{e}^{|v|_{\mathrm{L}}t} + 2 + \big|\tau_\partial(\cdot)\big|_{\mathrm{L}}\big)\big\|\mu_0 - \mu_0'\big\|_{\mathrm{BL}}^*.
\end{aligned}$$

The factor in front of $\|\mu_0 - \mu_0'\|_{\mathrm{BL}}^*$ is dominated by some C_T for $t \in [0,T]$. □

3. Approximation results

We consider a countable family of regularized systems defined by a decreasing sequence $(f_n) \subset \mathrm{BL}([0,1])$ of regularizers. Define $f_n(x) := [n(x-(1-\frac{1}{n}))]^+$, where $[\,\cdot\,]^+$ denotes the positive part of the argument. Denote by $\mu_t^{(n)}$ the mild solution to the regularized system defined by (3) with $F(\mu) = -af_n \cdot \mu$ for initial condition $\mu_0 \in \mathcal{M}^+([0,1])$. This perturbation is a bounded linear operator on $\overline{\mathcal{M}}([0,1])_{\mathrm{BL}}$ with $\|F\| \leqslant n+1$. Therefore, standard arguments yield the global existence, uniqueness and (Lipschitz) continuous dependence on initial conditions of *positive* measure-valued solutions.

A regularized solution can be viewed as describing the state of a system where there is absorption in a small layer (here of width $\frac{1}{n}$) at the boundary. We report here the following result:

Theorem 4. *Let $\mu_0 \in \mathcal{M}^+([0,1])$ be the initial datum. Then the sequence of solutions $(\mu^{(n)})$ to the regularized systems defined by (f_n) is a Cauchy sequence in the space $C([0,T], \mathcal{M}^+([0,1])_{\mathrm{BL}})$ for each $T > 0$. Moreover,*

$$\left\|\mu_t^{(n)} - \mu_t\right\|_{\mathrm{BL}}^* = \mathcal{O}\left(\frac{1}{n}\right) \quad \text{as } n \to \infty, \tag{9}$$

uniformly on compact time intervals.

The proofs of these results are given in Sections 4.2 and 4.3 of Ref. [5].

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