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Revisiting the Leinster groups

Quelques résultats sur les groupes de Leinster

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ABSTRACT

A finite group is said to be a Leinster group if the sum of the orders of its normal subgroups equals twice the order of the group itself. In this paper we give some new results concerning Leinster groups.

RÉSUMÉ

Un groupe de Leinster est un groupe fini tel que la somme des cardinaux de ses sous-groupes distingués soit égale au double du cardinal de G. Dans cette note, nous donnons quelques résultats nouveaux sur les groupes de Leinster.

1. Introduction

A number is perfect if the sum of its divisors equals twice the number itself. In 2001, T. Leinster [6], developed and studied a group-theoretic analogue of perfect numbers. A finite group is said to be a perfect group (not to be confused with the one which is equal to its commutator subgroup) or an immaculate group or a Leinster group if the sum of the orders of its normal subgroups equals twice the order of the group itself. Clearly, a finite cyclic group C_n is Leinster if and only if its order n is a perfect number. In fact, the abelian Leinster groups are precisely the finite cyclic groups whose orders are perfect numbers. It may be mentioned here that up to now, only one Leinster group of odd order is known, namely $(C_{127} \rtimes C_7) \times C_{3^4.11^2.19^2.113}$. It was discovered by F. Brunault [8]. More information on this and the related concepts can be found in the works of S.J. Baishya and A.K. Das [3], A.K. Das [4], M. Tărnăuceanu [10,12], T.D. Medts and A. Maróti [9], etc.

Given a finite group G, let $\tau(G)$ denote the number of normal subgroups of G and $\sigma(G)$ denote the sum of the orders of the normal subgroups of G. In this paper, among other results, we classify Leinster groups G with $\tau(G) \leqslant 7$.

2. Some basic results

The Leinster groups among the dihedral groups are in one-to-one correspondence with the odd perfect numbers [6, Example 2.4] and so it is an open question as to whether there are any. It would be interesting to find the Leinster groups among the well-known families of groups. In the following result, we classify the Leinster groups among the generalized quaternion group Q_{4m} of order $4m$, $m \geqslant 2$ given by $\langle a, b \mid a^{2m} = 1, b^2 = a^m, bab^{-1} = a^{-1}\rangle$.

Proposition 2.1. *The generalized quaternion group Q_{4m}, $m \geqslant 2$ is Leinster if and only if $m = 3$.*

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Proof. It is well known that $\frac{Q_{4m}}{Z(Q_{4m})} \cong D_{2m}$, for any integer $m \geqslant 2$. By [9, Observation 3.1], we have $\frac{\sigma(Q_{4m})}{|Q_{4m}|} \geqslant \frac{\sigma(D_{2m})}{|D_{2m}|}$. Now, if m is even, then by [6, Example 2.4], we have $\frac{\sigma(D_{2m})}{|D_{2m}|} > 2$ and so Q_{4m} is not Leinster. Next, suppose m is odd. In this situation, it can be easily proved that the proper normal subgroups of Q_{4m} are precisely the subgroups of the cyclic group generated by a. Consequently, $\sigma(Q_{4m}) = 4m + \sigma(2m)$, where $\sigma(2m)$ is the sum of the positive divisors of $2m$. We know that 6 is the only perfect number of the form $2m$, where m is odd. Therefore Q_{4m}, $m \geqslant 2$ is Leinster if and only if $m = 3$. $\square$

Let $I(G)$ denote the set of all solutions of the equation $x^2 = 1$ in G. If n is an odd perfect number, then D_{2n} is a Leinster group [6, Example 2.4] with $|I(D_{2n})| > n$. The following theorem shows that if G is a Leinster group such that $|I(G)| > \frac{|G|}{2}$, then $G \cong D_{2n}$, where n is an odd perfect number. In the following theorem $|Cent(G)|$ denotes the number of distinct centralizers of G. Recall that a finite group G is said to be a CA-group if the centralizer $C(x)$ is abelian for every $x \in G \setminus Z(G)$ (see [5]).

Theorem 2.2. *If G is a Leinster group with $|I(G)| > \frac{|G|}{2}$, then $G \cong D_{2n}$, where n is an odd perfect number.*

Proof. Let G be a Leinster group with $|I(G)| > \frac{|G|}{2}$. Clearly, $|G|$ is even, otherwise $|I(G)| = 1$ and hence G is trivial, which is not possible. Now, suppose G is abelian. Then G is cyclic by [6, Corollary 4.2] and hence $|I(G)| = 2$. It follows that $|G| = 2$, which is not possible. Therefore, G is non-abelian and without any loss we can assume that $|G| = 2^m n$, where $m \geqslant 1$ is an integer and $n > 1$ is an odd integer, noting that a 2-group is not Leinster [6, Example 2.3]. Let G' be the commutator subgroup of G. By [14, p. 251], we have $\frac{G}{G'}$ is an elementary abelian 2-group and therefore by [6, Theorem 4.1], we have $|\frac{G}{G'}| = 2$. Again, by [14, Theorem 5], we have G' is abelian and hence by [2, Theorem 2.3], we get:

$$\bigl|Cent(G)\bigr| = \bigl|G'\bigr| + 2 = \frac{|G|}{2} + 2. \tag{1}$$

It is easy to see that $Z(G) \subsetneq G'$. Now, it follows from (1) that the elements of $G \setminus G'$ will produce exactly $\frac{|G|}{2}$ distinct centralizers, noting that elements of G' produce exactly two distinct centralizers, namely G and $G' = C(g)$, where $g \in G' \setminus Z(G)$. Let $a, b \in G \setminus G'$ such that $a \neq b$. Since $|G \setminus G'| = \frac{|G|}{2}$, it follows that:

$$C(a) \neq C(b). \tag{2}$$

Again, since G' is an abelian normal subgroup of G of index 2, therefore by [5, Theorem A], we have G is a CA-group and hence by [5, Proposition 3.2], we get:

$$ab \neq ba. \tag{3}$$

Now, suppose $|Z(G)| \neq 1$. Let $x \in G \setminus G'$. Since $Z(G) \subsetneq C(x)$, therefore by (3), there exists $y \in G' \setminus Z(G)$ such that $y \in C(x)$. It follows that $x \in C(y) = G'$, which is a contradiction. Hence $|Z(G)| = 1$. Again, by [14, Lemma 9], we have $I(G') \subseteq Z(G)$ and hence $|G'| = n$, noting that $|\frac{G}{G'}| = 2$.

Let $z \in G \setminus G'$. If $|C(z)| \neq 2$, then there exists $w \in C(z)$ such that $w \neq z$ and $w \neq 1$. It follows from (3) that $w \in G' \setminus Z(G)$. But then $z \in C(w) = G'$, which is a contradiction. Therefore $|C(z)| = 2$ and hence:

$$\bigl|Cl(z)\bigr| = \frac{|G|}{2}. \tag{4}$$

Now, suppose $N \trianglelefteq G$, $N \neq G$. If $|N|$ is even, then there exists $u \in N$ such that $o(u) = 2$. Clearly, $u \in G \setminus G'$ and hence by (4), we have $|Cl(u)| = \frac{|G|}{2}$. But $Cl(u) \subsetneq N$, which is not possible. Therefore $|N|$ is odd and hence $N \trianglelefteq G'$.

Again, note that $v \sim v^{-1}$ for any $v \in G'$. For if, $v \nsim v^{-1}$ for some $v \in G'$ then the map:

$$\phi_v : I(G) \longrightarrow G \setminus I(G) \quad \text{given by } h \longmapsto hv$$

is one–one, which is not possible since $|I(G)| > \frac{|G|}{2}$.

Now, suppose $N \trianglelefteq G'$. Since G' is abelian of index 2, therefore $Cl(r) = \{r, r^{-1}\}$ for any $r \in G' \setminus Z(G)$. Hence $N \trianglelefteq G$. Therefore, the proper normal subgroups of G are precisely the normal subgroups of G'. In other words, $\sigma(G) = |G| + \sigma(G')$. Now, since G is a Leinster group, it follows that $|G| = 2|G'| = \sigma(G')$. Therefore by [6, Corollary 4.2], G' is cyclic and $|G'| = n$ is an odd perfect number. Hence $G \cong D_{2n}$. $\square$

A group is said to be semi-simple if it is a direct product of non-abelian simple groups. In this connection, we have the following result.

Proposition 2.3. *No finite semi-simple group is Leinster.*

Proof. Let G be a finite semi-simple group. Then $G = H_1 \times H_2 \times \cdots \times H_n$, where each H_i, $1 \leqslant i \leqslant n$ is a finite non-abelian simple group. By [4, Corollary 4.6], we have $\sigma(G) = \prod_i \sigma(H_i) = \prod_i (1 + |H_i|)$. Since finite non-abelian simple groups are of even order, $\sigma(G)$ is odd. Therefore, G is not Leinster. □

Continuing with the finite groups that are not Leinster, we also have the following result.

Proposition 2.4. *There is no Leinster group of order p^2q^2, where p, q are primes.*

Proof. Let G be a Leinster group of order p^2q^2, where p, q are primes. Since no p-group is Leinster [6, Example 2.3], therefore without any loss we may assume that $p < q$. Now, if G is abelian, then G is cyclic [6, Corollary 4.2] and hence $|G|$ is a perfect number, which is not possible.

Next, suppose that G is non-abelian. Now, if $q > 3$, then any q-Sylow subgroup of G is normal. Suppose G has no normal subgroup of order p. Then $p^2q^2 = 1 + qn$ for some positive integer n, which is not possible. Again, note that G can have at the most one normal subgroup of order p; otherwise the p-Sylow subgroup of G will be normal and G will be an abelian group, which is not possible. It follows that $p^2q^2 = 1 + p + qn$ for some positive integer n, which is again not possible, since $q > 3$. Therefore $q = 3$ and so $|G| = 36$, which is a contradiction by GAP [13]. □

We need the following remark for proving the next proposition.

Remark 2.5. Consider the group $G_m = \langle a, b \mid a^3 = b^{2^m} = 1, bab^{-1} = a^{-1}\rangle$. Using [9, Proposition 3.8], one can easily see that, if G_m is a Leinster group, then $m = 2$ and hence $G_m \cong Q_{12}$.

Proposition 2.6. *Let G be a Leinster group such that $\frac{G}{Z(G)} \cong S_3$. Then $G \cong S_3 \times C_5$ or $G \cong Q_{12}$.*

Proof. Let G be a Leinster group such that $\frac{G}{Z(G)} \cong S_3$. By [7, Corollary 2.2], we have $G = G_m \times A$, where $m \geqslant 1$ and A is an abelian group. Now, if $|A| = 1$, then $G \cong G_m$ and hence by Remark 2.5, $G \cong Q_{12}$. Next, suppose $|A| > 1$. We have $\langle a\rangle \times A \trianglelefteq G_m \times A$ and $|\frac{G_m \times A}{\langle a\rangle \times A}| = 2^m$. Suppose $m \geqslant 3$. Then by [9, Corollary 3.3], there exist $N_1 \trianglelefteq G$, $N_2 \trianglelefteq G$ and $N_3 \trianglelefteq G$ such that $|N_1| = \frac{|G|}{2}$, $|N_2| = \frac{|G|}{4}$ and $|N_3| = \frac{|G|}{8}$. It follows that $\sigma(G) > 2|G|$, which is impossible. Next, suppose $m = 2$. Then $G = Q_{12} \times A$. Note that Q_{12} have normal subgroups of order 2, 3 and 6, say N_1, N_2 and N_3, respectively. Therefore $\{1\} \times A, N_1 \times A$, $N_2 \times A$ and $N_3 \times A$ are normal subgroups of $Q_{12} \times A$ of indices $12, 6, 4$ and 2 respectively, which is again contradictory to the definition of Leinster groups. Therefore, $G = S_3 \times A$. In the present situation, one can verify that $\gcd(|S_3|, |A|) = 1$ and by [6, Corollary 3.2], $\sigma(S_3 \times A) = \sigma(S_3) \times \sigma(A)$. Now, since $G = S_3 \times A$ is Leinster, it follows that $6|A| = 5\sigma(A)$ and hence $A = C_5$. □

As an immediate corollary, we have the following result.

Corollary 2.7. *Consider the group $G = \langle a, b \mid a^{2n} = b^3 = 1, a^{-1}ba = b^{-1}\rangle$, $n \geqslant 1$. If G is Leinster, then $n = 2$ or 5 and hence $G \cong Q_{12}$ or $G \cong S_3 \times C_5$.*

Proof. By [1, Lemma 2.7], we have $|G| = 6n$ and $Z(G) = \langle a^2\rangle$. Therefore $\frac{G}{Z(G)} \cong S_3$ and by Proposition 2.6, we have $G \cong Q_{12}$ or $G \cong S_3 \times C_5$. □

3. Leinster groups with $\tau(G) \leqslant 7$

In this section, we classify Leinster groups with at the most seven normal subgroups. We begin with the following result which gives a necessary condition for a finite group to be Leinster.

Proposition 3.1. *Let G be a Leinster group. Then $\tau(G) \geqslant 4$, where the equality holds if and only if $G \cong C_6$.*

Proof. If $\tau(G) < 4$, then one can easily see that $\sigma(G) < 2|G|$. For the second part, suppose G is a Leinster group with $\tau(G) = 4$. Let M and N be the proper non-trivial normal subgroups of G. Let $|M| = m$ and $|N| = n$. Then we have:

$$|G| = 1 + m + n. \tag{5}$$

From (5), it follows that $G = MN$ and $M \cap N = \{1\}$. Without any loss, we may assume that $m < n$. Then, by (5), we have $m < n \leqslant m + 1$, forcing $n = m + 1$. Now, again using (5), we get $|G| = 6$ and hence $G \cong C_6$. The converse is trivial. □

As a consequence, we have the following result.

Corollary 3.2. *Let G be a finite group satisfying one of the following conditions*:

(a) $G \cong H \times K$, *where H and K are simple groups.*
(b) $|G| = pq$, *where p and q are primes.*

Then G is a Leinster group if and only if $G \cong C_6$.

Proof. Let G be a Leinster group satisfying one of the given conditions. Since p-groups are not Leinster [6, Example 2.3], the simple groups H and K (if (a) holds) are almost coprime in the sense of [4], and the primes p and q (if (b) holds) are distinct. Thus, if (a) is satisfied, then $\tau(G) = \tau(H)\tau(K) = 4$ (by [4, Corollary 4.6]), and so, $G \cong C_6$ (by Proposition 3.1). On the other hand, if (b) is satisfied, then $G \cong C_p \times C_q$ (otherwise, $\tau(G) = 3$), that is, G satisfies (a), and so we have $G \cong C_6$ once again. This completes the proof as C_6 is already seen to be Leinster. $\square$

As an immediate consequence, we also have the following result.

Corollary 3.3. *Let G be a finite group such that the proper non-trivial normal subgroups have the same order. Then G is not Leinster.*

Proof. In view of Proposition 3.1, we have $\tau(G) \geqslant 4$. Let M and N be two different normal subgroups of G having the same order. Then we have $M \cap N = \{1\}$ and $G = MN$. It follows that $G = M \times N$ and both M and N are simple groups. Now, the result follows using Corollary 3.2. $\square$

The following theorem classifies all Leinster groups with exactly five normal subgroups.

Theorem 3.4. *If G is a Leinster group with $\tau(G) = 5$, then $G \cong Q_{12}$.*

Proof. Let G be a Leinster group with $\tau(G) = 5$. Let N_1, N_2, N_3 be the proper non-trivial normal subgroups of G and n_1, n_2, n_3 denote their order respectively. Without any loss, we may assume that $1 < n_1 \leqslant n_2 \leqslant n_3$. From the definition of Leinster groups, we have:

$$|G| = 1 + n_1 + n_2 + n_3. \tag{6}$$

It follows using (6) that $n_3 \in \{\frac{|G|}{2}, \frac{|G|}{3}\}$. Now, suppose $n_3 = \frac{|G|}{3}$. Then by (6), we have:

$$2|G| = 3 + 3n_1 + 3n_2. \tag{7}$$

Clearly, $|N_1| \neq |N_2|$, otherwise $n_1 = n_2 = 3$, which is impossible. Now, if $N_1 \subsetneq N_2$, then by (7), we have $n_1 = 3$ and hence $n_2 = 6$ or 12 and consequently $|G| = 15$ or 24, which is a contradiction to the choice of G.

Next, suppose $N_1 \not\subseteq N_2$. Then $N_1 \cap N_2 = \{1\}$ and hence $N_1N_2 = N_3$, otherwise $G = N_1 \times N_2$, which implies $\tau(G) \neq 5$. Thus we have $n_1n_2 = n_3 = \frac{|G|}{3}$. It follows from (7) that $2n_1n_2 = 1 + n_1 + n_2$, which is again impossible. Therefore, we have $n_3 = \frac{|G|}{2}$ and by (6), we have:

$$|G| = 2 + 2n_1 + 2n_2. \tag{8}$$

In this situation also clearly $|N_1| \neq |N_2|$, otherwise $n_1 = n_2 = 2$, which is impossible. Now, if $N_1 \subsetneq N_2$, then by (8), we have $n_1 = 2$ and $n_2 = 6$, which is not possible to the choice of G. Therefore $N_1 \not\subseteq N_2$. It follows that $N_1 \cap N_2 = \{1\}$ and hence $N_1N_2 = N_3$, otherwise $G = N_1 \times N_2$, which implies $\tau(G) \neq 5$. Thus we have $n_1n_2 = n_3 = \frac{|G|}{2}$. Now, using (8) we get $n_1n_2 = 1 + n_1 + n_2$. It follows that $n_1 < n_2 \leqslant n_1 + 1$ and hence $n_2 = n_1 + 1$. Thus we get from (8) that $|G| = 4 + 4n_1$. Consequently, $n_1 = 2$ or $n_1 = 4$. Therefore, in view of the choice of G, it follows that $n_1 = 2$ and so $|G| = 12$. Now, it is a routine matter to see that $G \cong Q_{12}$. $\square$

We now classify Leinster groups with exactly six normal subgroups. We have used GAP [13] to verify some of the steps.

Theorem 3.5. *If G is a Leinster group with $\tau(G) = 6$, then $G \cong C_{28}$ or $S_3 \times C_5$.*

Proof. Let G be a Leinster group with $\tau(G) = 6$. Let N_1, N_2, N_3, N_4 be the proper non-trivial normal subgroups of G and n_1, n_2, n_3, n_4 denote their order respectively. Without any loss, we may assume that $1 < n_1 \leqslant n_2 \leqslant n_3 \leqslant n_4$. From the definition of Leinster groups, we have:

$$|G| = 1 + n_1 + n_2 + n_3 + n_4. \tag{9}$$

It follows from (9), that $n_4 \in \{\frac{|G|}{2}, \frac{|G|}{3}\}$. Now, suppose $n_4 = \frac{|G|}{3}$. Then again using (9), we have $n_3 \in \{\frac{|G|}{5}, \frac{|G|}{3}\}$, which is a contradiction to the choice of G. Therefore $n_4 = \frac{|G|}{2}$ and again using (9), we have $n_3 \in \{\frac{|G|}{4}, \frac{|G|}{5}, \frac{|G|}{6}, \frac{|G|}{7}\}$, noting that G

cannot have a normal subgroup of index 3. Now, if $n_3 = \frac{|G|}{7}$, then by (9), we have $n_2 > \frac{|G|}{7}$, which is a contradiction. Similarly, if $n_3 = \frac{|G|}{6}$, then also by (9), we end with a contradiction. Therefore $n_3 = \frac{|G|}{5}$ or $\frac{|G|}{4}$. Now, if $n_3 = \frac{|G|}{5}$, then by (9), we have $n_2 = \frac{|G|}{6}$ and so $|G| = 30$. Now, it is a routine mater to check that $G \cong S_3 \times C_5$. Next, suppose that $n_3 = \frac{|G|}{4}$. Then again using (9), we have $n_2 = \frac{|G|}{7}$. It follows that $|G| = 28$ and hence $G \cong C_{28}$. □

As a consequence we get the following result on Leinster groups of order pqr, where $p < q < r$ are primes. In the following result, G_n denotes the unique normal subgroup of order n, where n is a positive integer.

Theorem 3.6. *If G is a Leinster group of order pqr, where $p < q < r$ are primes, then $G \cong S_3 \times C_5$.*

Proof. One can easily verify that every normal subgroup of G is uniquely determined by its order. It follows using Proposition 3.1 and Theorem 3.4 that $6 \leqslant \tau(G) \leqslant 8$. If $\tau(G) = 8$, then $G \cong C_{pqr}$. Therefore pqr is a perfect number, which is impossible (see [11]). Next suppose that $\tau(G) = 7$. Note that by Sylow theorem, we already have four normal subgroups, namely G_1, G_{pqr}, G_r and G_{qr}. Now, if $G_p \trianglelefteq G$ and $G_q \trianglelefteq G$, then $G \cong C_{pqr}$ and $\tau(G) = 8$, which is a contradiction. Therefore G_{pq} and G_{pr} must be normal in G. But then $G_{pq} \cap G_{pr} = G_p \trianglelefteq G$ and $G_{pq} \cap G_{qr} = G_q \trianglelefteq G$, which is again a contradiction. Therefore we have $\tau(G) = 6$. Now, the result follows using Theorem 3.5. □

We conclude this section with the following result on Leinster groups with exactly seven normal subgroups.

Theorem 3.7. *If G is a Leinster group with $\tau(G) = 7$, then $G \cong C_7 \rtimes C_8$.*

Proof. Let G be a Leinster group with $\tau(G) = 7$. Let N_1, N_2, N_3, N_4, N_5 be the proper non-trivial normal subgroups of G and n_1, n_2, n_3, n_4, n_5 denote their order respectively. Without any loss, we may assume that $1 < n_1 \leqslant n_2 \leqslant n_3 \leqslant n_4 \leqslant n_5$. From the definition of Leinster groups, we have:

$$|G| = 1 + n_1 + n_2 + n_3 + n_4 + n_5. \tag{10}$$

In view of (10), using GAP [13] and certain standard results from the theory of finite groups, we can see that $n_i \notin \{\frac{|G|}{3}, \frac{|G|}{5}\}$ for any $i \in \{1, 2, 3, 4, 5\}$ and $n_i = \frac{|G|}{4}$ for some $i \in \{1, 2, 3, 4, 5\}$. Therefore, by correspondence theorem, $n_5 = \frac{|G|}{2}$ and $n_4 = \frac{|G|}{4}$. Now, (10) becomes:

$$|G| = 4(1 + n_1 + n_2 + n_3). \tag{11}$$

Since $n_3 \notin \{\frac{|G|}{3}, \frac{|G|}{5}\}$, it follows from (11) that $n_3 \in \{\frac{|G|}{14}, \frac{|G|}{13}, \frac{|G|}{12}, \frac{|G|}{11}, \frac{|G|}{10}, \frac{|G|}{8}, \frac{|G|}{7}, \frac{|G|}{6}\}$. In view of (11), again using GAP [13] and certain standard results from the theory of finite groups, we can see that $n_3 = \frac{|G|}{8}$. Therefore (11) becomes:

$$|G| = 8(1 + n_1 + n_2). \tag{12}$$

Since $n_2 \notin \{\frac{|G|}{3}, \frac{|G|}{5}\}$, it follows from (12) that $n_2 \in \{\frac{|G|}{23}, \frac{|G|}{22}, \frac{|G|}{20}, \frac{|G|}{19}, \frac{|G|}{18}, \frac{|G|}{17}, \frac{|G|}{16}, \frac{|G|}{14}, \frac{|G|}{13}, \frac{|G|}{11}, \frac{|G|}{10}\}$. In view of (12), again using GAP [13] and certain standard results from the theory of finite groups, we can see that $n_2 = \frac{|G|}{14}$. Therefore (12) becomes

$$3|G| = 56(1 + n_1). \tag{13}$$

From (13) it follows that $n_1 \mid 56$. Now, if $n_1 = 56$, then $|G| = 2^3.7.19$. But then the Sylow 19-subgroup of G is normal in G, which is not possible. Next, if $2 < n_1 < 56$, then in view of (13), using GAP [13], we get a contradiction. Hence $n_1 = 2$ and by (13) we get $|G| = 56$. Therefore using GAP [13], we have $G \cong C_7 \rtimes C_8$. □

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References

[1] A.R. Ashrafi, Counting the centralizers of some finite groups, Korean J. Comput. Appl. Math. 7 (1) (2000) 115–124.
[2] S.J. Baishya, On finite groups with specific number of centralizers, Int. Electron. J. Algebra 13 (2013) 53–62.
[3] S.J. Baishya, A.K. Das, Harmonic numbers and finite groups, Rend. Semin. Mat. Univ. Padova (2013), in press, http://rendiconti.math.unipd.it/forthcoming.php?lan=english#BaishyaDas.
[4] A.K. Das, On arithmetic functions of finite groups, Bull. Aust. Math. Soc. 75 (2007) 45–58.
[5] S. Dolfi, M. Herzog, E. Jabara, Finite groups whose non-central commuting elements have centralizers of equal size, Bull. Aust. Math. Soc. 82 (2010) 293–304.
[6] T. Leinster, Perfect numbers and groups, arXiv:math.GR/0104012v1, April 2001.

[7] P. Lescot, Central extensions and commutativity degree, Comm. Algebra 29 (10) (2001) 4451–4460.
[8] MathOverflow, http://mathoverflow.net/questions/54851.
[9] T.D. Medts, A. Maróti, Perfect numbers and finite groups, Rend. Semin. Mat. Univ. Padova 129 (2013) 17–33.
[10] T.D. Medts, M. Tărnăuceanu, Finite groups determined by an inequality of the orders of their subgroups, Bull. Belg. Math. Soc. Simon Stevin 15 (4) (2012) 699–704.
[11] O. Ore, On the averages of the divisors of a number, Amer. Math. Monthly 55 (1948) 615–619.
[12] M. Tărnăuceanu, Finite groups determined by an inequality of the orders of their normal subgroups, An. Ştiinţ. Univ. "Al.I. Cuza" Iaşi, Mat. 57 (2011) 229–238.
[13] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.6.4, http://www.gap-system.org, 2013.
[14] C.T.C. Wall, On groups consisting mostly of involutions, Proc. Camb. Philos. Soc. 67 (1970) 251–262.