C. R. Acad. Sci. Paris, Ser. I 351 (2013) 753–759

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Functional analysis

Honesty in discrete, nonlocal and randomly position structured fragmentation model with unbounded rates

Honnêteté dans un modèle discret non local de fragmentation structurée aléatoire en espace dans le cas de taux non bornés

Emile Franc Doungmo Goufo, Suares Clovis Oukouomi Noutchie

Department of Mathematical Sciences, North-West University, Mafikeng, South Africa

A R T I C L E   I N F O

Article history:
Received 28 May 2013
Accepted after revision 30 September 2013
Available online 21 October 2013

Presented by the Editorial Board

A B S T R A C T

In the process of discrete and nonlocal aggregation, the major problem arises when each fragmentation rate becomes infinite at infinity. In this paper, a discrete Cauchy problem describing multiple fragmentation processes is investigated by means of parameter-dependent operators together with the theory of substochastic semigroups with a parameter. We focus on the case where fragmentation rates are size and position dependent and where new particles are spatially randomly distributed according to a certain probabilistic law. Unlike [8], where the discrete model with bounded fragmentation rates is treated, we use, in this paper, Kato's theorem in L_1 [2] and the dominated convergence theorem [4] to show the existence of the infinitesimal generator of a positive semigroup of contractions and give sufficient conditions for honesty in the case of unbounded fragmentation rates. Essentially, we demonstrate that, even in the discrete and nonlocal case, the process is conservative if at infinity daughter particles tend to go back into the system with a high probability.

R É S U M É

Dans un processus d'agrégation discret non local, un problème fondamental se pose lorsque chaque taux de fragmentation tend vers l'infini à l'infini. Dans cette Note, on étudie le problème de Cauchy discret dans le cas où les taux de fragmentation décrivent des processus de fragmentation multiple au moyen d'opérateurs dépendant de paramètres et de la théorie des semi-groupes sous-stochastiques dépendant d'un paramètre. On se concentre sur le cas où les taux de fragmentation dépendent de la dimension et de la position et où de nouvelles particules sont distribuées de manière aléatoire suivant une certaine loi de probabilité. À la différence de [8], qui traite d'un modèle discret à taux de fragmentation borné, on utilise le théorème de Kato dans le cas L_1 [2] et le théorème de la convergence dominée [4] pour démontrer l'existence d'un générateur infinitésimal d'un semi-groupe de contactions positif ; on donne des conditions suffisantes d'honnêteté dans le cas de taux de fragmentation non bornés. Fondamentalement, on démontre que, même dans le cas discret et non local, le processus est conservatif si, à l'infini, les particules filles tendent à rentrer dans le système avec une grande probabilité.

E-mail address: franckemile2006@yahoo.ca (E.F. Doungmo Goufo).

1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2013.09.023

1. Introduction

Fragmentation processes can be observed in natural sciences and engineering. To provide just a few examples, we mention the study of stellar fragments in astrophysics, rock fracture, degradation of large polymer chains, DNA fragmentation, evolution of phytoplankton aggregates, liquid droplet break-up or break-up of solid drugs in organisms. We also have external processes such as oxidation, melting, or dissolution, which cause the exposed surface of particles to recede, resulting in the fragmentation with loss of mass. There exists a vast literature on fragmentation equations and many of them have been deeply analyzed in different works (see, e.g., [2,3,5,7,9–11]). Kinetic-type models with diffusion were investigated in [1], where the author showed that the diffusive part does not affect the breach of the conservation laws. But discrete fragmentation processes have not widely been investigated yet. In [3], a discrete model with the concentration depending only on the size n of clusters and time t is analyzed, and the author used compactness of the semigroups to analyze their long-time behavior and concluded that they have the asynchronous growth property. Conservative and nonconservative fragmentation processes have been thoroughly investigated, and, in particular, the breach of the mass conservation law (called shattering) has been attributed to a phase transition creating a dust of "zero-size" particles with nonzero mass, which are beyond the resolution of the model. Shattering can be interpreted from an analytic point of view as dishonesty of the semigroup associated with the model [2], and from the probabilistic point of view as the outburst in the Markov process describing the fragmentation [6,9].

When the fragmentation rate depends on both size and position, the new particles resulting from fragmentation can be spatially randomly distributed according to a certain probability density. If we assume that every group of size $n \in \mathbb{N}$ (one n-group) in a system of particles clusters consists of n identical fundamental units (monomers), then its total mass is simply a multiple positive integer of the mass of the monomers. In this work, we consider only discrete clusters; that is, they consist of a finite number of elementary (unbreakable) particles that are assumed to be of unit mass. The evolution of such particle–mass–position distribution can be derived by balancing loss and gain of clusters of size n (with position x) over a short period of time and is given by the following Cauchy problem [3]:

$$\frac{\partial p}{\partial t}(t,x,n) = -a_n(x)p(t,x,n) + \sum_{m=n+1}^{\infty} \int_{\mathbb{R}^3} a_m(y)b_{n,m}(y)h(x,n,m,y)p(t,y,m)\,\mathrm{d}y, \quad n = 1,2,3,\dots \tag{1}$$

with the initial mass-position distribution:

$$p(0,x,n) = \mathring{p}_n(x), \quad n = 1,2,3,\dots, \tag{2}$$

where in terms of n and x, the state of the system is characterized at any moment t by the *density* (or *concentration*) of particles $p(t,x,n)$.

In [8], the authors exploited a technique called the method of semigroups with a parameter [2] to analyze discrete fragmentation models with bounded fragmentation rates and concentration of particles depending not only on the size n of clusters and time t, but also on the random position x of the clusters in the space. In this paper, we follow the same method and extend the results to the case where the fragmentation rate $a_n(x)$ becomes infinite as $|x|$ is close to infinity.

2. Models' description and assumptions

As said earlier we assume that the mass n of a particle takes its values in $\mathbb{N}$. The particle–mass–position distribution $p : \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{N} \to \mathbb{R}_+$ will be noted $p(t,x,n) = p_n(t,x) \equiv p_n$ for the sake of simplicity. The rate $a(x,n) = a_n(x) \equiv a_n$ represents the fraction of groups of size n undergoing break-up at position x during the unit time. Because aggregates of size one cannot split, we assume that:

$$a_1(x) = 0 \tag{3}$$

for every $x \in \mathbb{R}^3$. After a group's fragmentation, new originating daughter particles have different centers distributed according to a given probabilistic law $h(\cdot,n,m,y)$. This is the probability density that, after a fragmentation of an m-aggregate (with the center at y), the new formed n-group will be located at the position x. Therefore,

$$\int_{\mathbb{R}^3} h(x,n,m,y)\,\mathrm{d}x = 1. \tag{4}$$

When an m-aggregate located at x breaks, the expected average number of n-group produced upon the beak-up is a non-negative measurable function $b_{n,m}(x) = b(x,n,m)$ defined on $\mathbb{R}^3 \times \mathbb{N} \times \mathbb{N}$. Since a group of size $m \leqslant n$ cannot split to form a group of size n, then $\mathrm{Supp}(b) \subseteq \mathbb{R}^3 \times \{(n,m) \in \mathbb{N} \times \mathbb{N}:\ m > n\}$, which yields:

$$b_{n,m} = 0 \quad \text{for all } m \leqslant n. \tag{5}$$

Moreover, the sum of all individuals obtained by fragmentation of an n-group should again be n, hence it follows that for any $n \in \mathbb{N}$, $x \in \mathbb{R}^3$:

$$\sum_{m=1}^{n-1} mb(x,m,n) = n. \tag{6}$$

3. Well-posedness of the fragmentation problem

Since the total number of particles is expected to be conserved, it is appropriate to work in the Banach space:

$$\mathcal{X}_1 := \left\{ \mathbf{g} = (g_n)_{n=1}^{\infty}\colon \mathbb{R}^3 \times \mathbb{N} \ni (x,n) \to g_n(x),\ \|\mathbf{g}\|_1 := \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n\big|g_n(x)\big|\,\mathrm{d}x < \infty \right\},$$

because the norm of its elements represents the total mass (or total number of individuals) of the system and must be finite. In $\mathcal{X}_1$ we recast (1) and (2) in more compact form,

$$\frac{\partial}{\partial t}\mathbf{p} = \mathfrak{A}\mathbf{p} + \mathfrak{B}\mathbf{p}, \quad \mathbf{p}_{|t=0} = \mathring{\mathbf{p}}. \tag{7}$$

Here $\mathbf{p}$ is the vector $(p(t,x,n))_{n\in\mathbb{N}}$, $\mathfrak{A}$ is the diagonal matrix $(a_n(x))_{n\in\mathbb{N}}$, $\mathfrak{B}$ is defined by the expression:

$$\mathfrak{B}\mathbf{p} = \left(\sum_{m=n+1}^{\infty} \int_{\mathbb{R}^3} a_m(y) b_{n,m}(y) h(x,n,m,y) p_m(y)\,\mathrm{d}y \right)_{n=1}^{\infty}, \tag{8}$$

$\mathring{\mathbf{p}}$ being the initial vector $(\mathring{p}_n(x))_{n\in\mathbb{N}}$, which belongs to $\mathcal{X}_1$. We introduce operators $\mathbf{A}$ and $\mathbf{B}$ defined in $\mathcal{X}_1$ by:

$$\begin{aligned} &[\mathbf{A}\mathbf{p}](x,n) = [\mathfrak{A}\mathbf{p}](x,n), && D(\mathbf{A}) = \{\mathbf{g} \in \mathcal{X}_1;\ a\mathbf{g} \in \mathcal{X}_1\}; \\ &[\mathbf{B}\mathbf{p}](x,n) = [\mathfrak{B}\mathbf{p}](x,n), && D(\mathbf{B}) := D(\mathbf{A}). \end{aligned} \tag{9}$$

Lemma 3.1. *The operator sum* $(\mathbf{A}+\mathbf{B}, D(\mathbf{A}))$ *is well defined.*

Proof. It suffices to show that $\mathbf{B}D(\mathbf{A}) \subset \mathcal{X}_1$. For every $\mathbf{g} \in D(\mathbf{A})$,

$$\begin{aligned} \|\mathbf{B}\mathbf{g}\|_1 &= \int_{\mathbb{R}^3} \left(\sum_{n=1}^{\infty} n \sum_{m=n+1}^{\infty} \int_{\mathbb{R}^3} a_m(y) b_{n,m}(y) h(x,n,m,y)\big|g(y,m)\big|\,\mathrm{d}y \right) \mathrm{d}x \\ &= \int_{\mathbb{R}^3} \left(\sum_{n=1}^{\infty} n \sum_{m=n+1}^{\infty} a_m(y) b_{n,m}(y)\big|g(y,m)\big| \right) \mathrm{d}y \\ &= \int_{\mathbb{R}^3} \sum_{m=2}^{\infty} a_m(y)\big|g(y,m)\big| \left(\sum_{n=1}^{\infty} n b_{n,m}(y) \right) \mathrm{d}y \\ &= \int_{\mathbb{R}^3} \sum_{m=2}^{\infty} a_m(y)\big|g(y,m)\big| \left(\sum_{n=1}^{m-1} n b_{n,m}(y) \right) \mathrm{d}y \\ &= \int_{\mathbb{R}^3} \sum_{m=2}^{\infty} m a_m(y)\big|g(y,m)\big|\,\mathrm{d}y \\ &= \int_{\mathbb{R}^3} \sum_{m=1}^{\infty} m a_m(y)\big|g(y,m)\big|\,\mathrm{d}y = \|\mathbf{A}\mathbf{g}\|_1 < \infty, \end{aligned}$$

where we have used (4), (6) and (5) respectively. Then $\|\mathbf{B}\mathbf{g}\|_1 = \|\mathbf{A}\mathbf{g}\|_1$, $\forall \mathbf{g} \in D(\mathbf{A})$, so that we can take $D(\mathbf{B}) := D(\mathbf{A})$ and the assertion follows. □

3.1. Mathematical setting and analysis

We note that the operators $\mathbf{A}$ and $\mathbf{B}$ defined in (9) have the property that one of the variables is a parameter and, for each value of this parameter, the operator has a certain desirable property (like being the generator of a semigroup) with respect to the other variable. Thus we need to work with parameter-dependent operators that can be "glued" together in such a way that the resulting operator inherits the properties of the individual ones. Next we provide more details about this technique, called the method of semigroups with a parameter [2].

Let $\Lambda = \mathbb{R}^3 \times \mathbb{N}$ and consider the space $\mathcal{X} := L_p(V, X)$ where $1 \leqslant p < \infty$, $(V, \mathrm{d}m)$ is a measure space and X a Banach space. Let us suppose that we are given a family of operators $\{(A_v, D(A_v))\}_{v\in V}$ in X and define the operator $(\mathbb{A}, D(\mathbb{A}))$ acting in $\mathcal{X}$ according to the following formulae,

$$\mathcal{D}(\mathbb{A}) := \big\{g \in \mathcal{X};\ g(v) \in D(A_v) \text{ for almost every } v \in V,\ \mathbb{A}g \in \mathcal{X}\big\}, \tag{10}$$

and, for $g \in \mathcal{D}(\mathbb{A})$,

$$(\mathbb{A}g)(v) := A_v g(v), \tag{11}$$

for every $v \in \mathcal{X}_1$. We have the following proposition.

Proposition 3.2. *(See [2], Proposition 3.28.) If for almost any $v \in V$ the operator A_v is m-dissipative in X, and the function $v \longrightarrow R(\lambda, A_v)g(v)$ is measurable for any $\lambda > 0$ and $g \in \mathcal{X}$, then the operator $\mathbb{A}$ is an m-dissipative operator in $\mathcal{X}$. If $(G_v(t))_{t\geqslant 0}$ and $(\mathcal{G}(t))_{t\geqslant 0}$ are the semigroups generated by A_v and $\mathbb{A}$, respectively, then for almost every $v \in V$, $t \geqslant 0$, and $g \in \mathcal{X}$, we have*

$$\big[\mathcal{G}(t)g\big](v) := G_v(t)g(v). \tag{12}$$

Making use of the above ideas, we introduce relevant operators for the present application. We take the variable n as the parameter and x as the main variable. We set:

$$X_x := L_1\big(\mathbb{R}^3, \mathrm{d}x\big) := \left\{\psi\colon\ \|\psi\| = \int_{\mathbb{R}^3} \big|\psi(x,n)\big|\,\mathrm{d}x < \infty\right\}$$

and define in X_x the operators $(\mathcal{A}_n, D(\mathcal{A}_n))$ as

$$\mathcal{A}_n p(t,x,n) = a_n(x)p(t,x,n) \quad \text{and} \quad D(\mathcal{A}_n) := \{p_n \in X_x, \mathcal{A}_n p_n \in X_x\}, \quad n \in \mathbb{N}. \tag{13}$$

Using Proposition 3.2, we can take $\mathbb{A} = \mathbf{A}$, $\mathcal{X} = \mathcal{X}_1 = L_1(\mathbb{N}, X_x) = L_1(\Lambda, \mathrm{d}\mu\,\mathrm{d}\varsigma) = L_1(\mathbb{R}^3 \times \mathbb{N}, \mathrm{d}\mu\,\mathrm{d}\varsigma)$, where $\mathbb{N}$ is equipped with the counting measure $\mathrm{d}\varsigma$ and $\mathrm{d}\mu = \mathrm{d}x$ is the Lebesgue measure in $\mathbb{R}^3$. In the notation of the proposition, $(\mathbb{N}, \mathrm{d}\varsigma) = (V, \mathrm{d}m)$, $X_x = X$ and $A_v = \mathcal{A}_n$, therefore $(\mathcal{A}_n, D(\mathcal{A}_n))_{n\in\mathbb{N}}$ is a family of operators in X_x and using (11), we have:

$$(\mathbf{Ap})_n := \mathcal{A}_n p_n. \tag{14}$$

Theorem 3.3. *There is an extension K of $\mathbf{A} + \mathbf{B}$ that generates a positive semigroup of contractions $(S_K(t))_{t\geqslant 0}$ on $\mathcal{X}_1$. Moreover, for each $\mathring{\mathbf{p}} = (\mathring{p}_n(x))_{n\in\mathbb{N}} \in D(K)$, there is a measurable representation $\mathbf{p}$ of $S_K(t)\mathring{\mathbf{p}}$ that is absolutely continuous with respect to $t \geqslant 0$ for almost any (x, n) and such that (7) is satisfied almost everywhere.*

Proof. See [8, Theorem 3.3]. □

In general, for each $n \in \mathbb{N}$, the function $G_{K_n}(t)\mathring{p}_n$ is not differentiable if $\mathring{p}_n \in X_x \backslash D(K_n)$. Therefore, it cannot be a classical solution of the Cauchy equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} G_{K_n}(t)\mathring{p}_n = K_n G_{K_n}(t)\mathring{p}_n, \tag{15}$$

where the equality holds for any $t > 0$ in the sense of equality in X_x. The initial condition is satisfied in the following sense:

$$\lim_{t\to 0^+} G_{K_n}(t)\mathring{p}_n = \mathring{p}_n, \tag{16}$$

where the convergence is in the X_x-norm. However, it is a mild solution, that is, it is a continuous function such that $\int_0^t p_n(\tau)\,\mathrm{d}\tau \in D(K_n)$ for any $t \geqslant 0$, satisfying the integrated version of (15), (16):

$$p_n(t) = \mathring{p}_n + K_n \int_0^t p_n(\tau)\,\mathrm{d}\tau. \tag{17}$$

Corollary 3.4. *If $\mathring{p}_n \in X_x \backslash D(K_n)$, then $p_n = [G_{K_n}(t)\mathring{p}_n](x, n)$ satisfies the equation:*

$$p(t,x,n) = \mathring{p}_n(x,n) - a_n(x)\int_0^t p(\tau,x,n)\,\mathrm{d}\tau + \sum_{m=n+1}^{\infty} \int_{\mathbb{R}^3} a_m(y)b_{n,m}(y)h(x,n,m,y)p_m(y)\left(\int_0^t p(\tau,y,n)\,\mathrm{d}\tau\right)\mathrm{d}y.$$

Proof. See [8, Corollary 3.4]. □

In the next section we provide a fairly general condition for honesty of $(G_{K_n}(t))_{t\geqslant 0}$.

4. Honesty

As stated in the introduction, the conservation of total mass is not always satisfied in the system. In fact, by analyzing models with specific coefficients, several authors have observed that the local version of the conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t)=0 \tag{18}$$

is not valid [11], where $U(t)=\sum_{n=1}^{\infty}\int_{\mathbb{R}^3} np(t,x,n)\,\mathrm{d}x=\sum_{n=1}^{\infty} n\int_{\mathbb{R}^3} p(t,x,n)\,\mathrm{d}x$ is the total number of particles (total mass) in the system. In other words, there occurs an unexpected mass loss in the system. In local fragmentation, the unaccounted for mass loss was termed *shattering fragmentation* and was attributed to the phase transition in which a dust of particles with zero size and nonzero mass is formed. The presence of x in (18) suggests that honesty in nonlocal discrete fragmentation depends also on the spatial properties of the fragmentation kernels. However the fragmentation process itself does not modify the total number of individuals in a population and therefore the law (18) is supposed to be satisfied throughout the evolution. This is formally expressed by (1), as the mass rate equation can be found by multiplying (1) by n, integrating over $\mathbb{R}^3$, summing from $n=1$ to ∞ and using (6), which agrees with the physics of the process, as fragmentation should simply rearrange the distribution of masses of the particles without altering the total mass of the system. However, the validity of (18) depends on certain properties of the solution p that we tacitly assumed during the integration and are far from obvious. In the following lines, we provide sufficient conditions for the discrete fragmentation semigroup to be honest for general coefficients.

Lemma 4.1. *Assume that for any* $\mathbf{p}=(p_n)_{n=1}^{\infty}\in(\mathcal{X}_1)_+$ *such that* $-\mathbf{Ap}+\mathbf{Bp}\in\mathcal{X}_1$*, we have the inequality*:

$$\int\limits_{\Lambda}(-\mathbf{Ap}+\mathbf{Bp})\,\mathrm{d}\mu\,\mathrm{d}\varsigma\geqslant 0, \tag{19}$$

then $K=\overline{\mathbf{A}+\mathbf{B}}$*. Thus the solution* $(p_n)_{n=1}^{\infty}=\mathbf{p}=G_K(t)\mathring{\mathbf{p}}=(G_{K_n}(t)\mathring{p}_n)_{n=1}^{\infty}$ *satisfies*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=1}^{\infty}\int\limits_{\mathbb{R}^3}G_{K_n}(t)\mathring{p}_n(x,n)n\,\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}t}\left\|G_{K_n}(t)\mathring{p}_n\right\|=0$$

and for any $\mathring{\mathbf{p}}_n=(\mathring{p})_{n=1}^{\infty}\in D(K)_+$*. In other words, the semigroup* $(G_K(t))_{t\geqslant 0}$ *is honest.*

Proof. See [8, Lemma 4.1]. □

Now we assume that for each $n\in\mathbb{N}$, there are two constants $0<\theta_1$ and θ_2 such that:

$$\theta_1\alpha_n\leqslant a_n(x)\leqslant\theta_2\alpha_n, \tag{20}$$

with $\alpha_n\in\mathbb{R}_+$ and independent of the state variable x. The previous lemma allows us to state the following theorem:

Theorem 4.2. *Assume that the condition (20) above is satisfied for almost all* $(x,n)\in\mathbb{R}^3\times\mathbb{N}$*, then the semigroup* $(G_K(t))_{t\geqslant 0}$ *is honest.*

Proof. See [8, Theorem 4.2]. □

The previous theorem shows that when each discrete fragmentation rate a_n is bounded by a size-only dependent function, the spatial and random distribution of the particles has no influence on the conservativeness of the system. In other words nonlocal discrete models with each $a_n(x)$ bounded as $|x|$ approaches infinity always behave like local models, therefore are conservative provided that the fragmentation rate a_n is bounded as n approaches zero. However, there is a major complication [2] that arises when, in the discrete case, each fragmentation rate $a_n(x)$ becomes infinite as $|x|$ is close to infinity. The next theorem gives sufficient conditions for conservativeness in that case.

Theorem 4.3. *Assume that for each* $n\in\mathbb{N}$*, we have*:

$$a_n\in L_{\infty,loc}\left(\mathbb{R}^3\right) \tag{21}$$

and there exists $K > 0$ such that:

$$a_m(y) \int\limits_{|x|>|y|} h(x,n,m,y)\,\mathrm{d}x < K \tag{22}$$

is satisfied for almost all $(x,m) \in \mathbb{R}^3 \times \mathbb{N}$, then the semigroup $(G_K(t))_{t\geqslant 0}$ is honest.

Proof. The proof is based on [2, Theorem 6.13]. Let $\mathbf{p} = (p_n)_{n=1}^{\infty} \in (\mathcal{X}_1)_+$, by (21), for any $0 < N_1 < \infty$ we have that $a_n p_n \in L_1(B(O,N_1), n\,\mathrm{d}x)$, where $B(O,N_1)$ represents the ball $\{x \in \mathbb{R}^3; |x| \leqslant N_1\}$. Because $-\mathbf{Ap} + \mathbf{Bp} \in \mathcal{X}_1$, we also have $\mathcal{B}_n p_n \in L_1(B(O,N_1), n\,\mathrm{d}x)$. So, making use of Lemma 4.1, it is enough to prove that the inequality $\int_\Lambda (-\mathbf{Ap} + \mathbf{Bp})\,\mathrm{d}\mu\,\mathrm{d}\varsigma \geqslant 0$ is satisfied. We have:

$$\int\limits_\Lambda (-\mathbf{Ap} + \mathbf{Bp})\,\mathrm{d}\mu\,\mathrm{d}\varsigma = \sum_{n=1}^{\infty} \int\limits_{\mathbb{R}^3} \big(-a(x,n)p_n(x) + [\mathcal{B}_n p_n](x)\big) n\,\mathrm{d}x$$
$$= \lim_{N,N_1\to\infty} \left(\sum_{n=1}^{N} \int\limits_{B(O,N_1)} -a(x,n)p_n(x) n\,\mathrm{d}x + \sum_{n=1}^{N} \int\limits_{B(O,N_1)} [\mathcal{B}_n p_n](x) n\,\mathrm{d}x \right).$$

We have

$$\sum_{n=1}^{N} \int\limits_{B(O,N_1)} [\mathcal{B}_n p_n](x) n\,\mathrm{d}x = \sum_{n=1}^{N} \int\limits_{B(O,N_1)} \left(\sum_{m=n+1}^{\infty} \int\limits_{\mathbb{R}^3} a_m(y) b_{n,m}(y) \big(h(x,n,m,y)\big) p_m(y)\,\mathrm{d}y \right) n\,\mathrm{d}x$$
$$= Q(N,N_1) + \sum_{m=1}^{N} \int\limits_{\mathbb{R}^3} \sum_{n=1}^{m-1} \int\limits_{B(O,N_1)} a_m(y) b_{n,m}(y) h(x,n,m,y) p_m(y) n\,\mathrm{d}x\,\mathrm{d}y,$$

where

$$Q(N,N_1) = \sum_{m=N+1}^{\infty} \int\limits_{\mathbb{R}^3} \sum_{n=1}^{N} \int\limits_{B(O,N_1)} a_m(y) b_{n,m}(y) h(x,n,m,y) p_m(y) n\,\mathrm{d}x\,\mathrm{d}y \geqslant 0,$$

with h defined by (4). It follows that:

$$\sum_{n=1}^{N} \int\limits_{B(O,N_1)} [\mathcal{B}_n p_n](x) n\,\mathrm{d}x \geqslant \sum_{m=1}^{N} \int\limits_{\mathbb{R}^3} a_m(y) p_m(y) \left(\sum_{n=1}^{m-1} \int\limits_{B(O,N_1)} b_{n,m}(y) h(x,n,m,y) n\,\mathrm{d}x \right) \mathrm{d}y$$
$$\geqslant \sum_{m=1}^{N} \int\limits_{B(O,N_1)} a_m(y) p_m(y) \left(\sum_{n=1}^{m-1} \int\limits_{B(O,N_1)} b_{n,m}(y) h(x,n,m,y) n\,\mathrm{d}x \right) \mathrm{d}y.$$

Thus:

$$\sum_{n=1}^{N} \int\limits_{B(O,N_1)} [\mathcal{B}_n p_n](x) n\,\mathrm{d}x$$
$$\geqslant \sum_{m=1}^{N} \int\limits_{B(O,N_1)} a_m(y) p_m(y) m\,\mathrm{d}y - \sum_{m=1}^{N} \int\limits_{B(O,N_1)} a_m(y) p_m(y) \left(\sum_{n=1}^{m-1} \int\limits_{|x|>N_1} b_{n,m}(y) h(x,n,m,y) n\,\mathrm{d}x \right) \mathrm{d}y.$$

Hence

$$\sum_{n=1}^{N} \int\limits_{B(O,N_1)} \big(-a(x,n)p_n(x)\big) n\,\mathrm{d}x + \sum_{n=1}^{N} \int\limits_{B(O,N_1)} [\mathcal{B}_n p_n](x) n\,\mathrm{d}x$$
$$\geqslant -\sum_{m=1}^{N} \int\limits_{B(O,N_1)} a_m(y) p_m(y) \left(\sum_{n=1}^{m-1} \int\limits_{|x|>N_1} b_{n,m}(y) h(x,n,m,y) n\,\mathrm{d}x \right) \mathrm{d}y.$$

By the assumption (22), for any $y \in B(O, N_1)$, we have:

$$a_m(y) \int\limits_{|x|>N_1} h(x,n,m,y)\,\mathrm{d}x \leqslant a_m(y) \int\limits_{|x|>|y|} h(x,n,m,y)\,\mathrm{d}x < K.$$

Using (6), this implies that:

$$\sum_{m=1}^{\infty} \int\limits_{B(O,N_1)} a_m(y)p_m(y)\left(\sum_{n=1}^{m-1} \int\limits_{|x|>N_1} b_{n,m}(y)h(x,n,m,y)n\,\mathrm{d}x\right)\mathrm{d}y$$

$$\leqslant K \sum_{m=1}^{\infty} \int\limits_{\mathbb{R}^3} p_m(y)\left(\sum_{n=1}^{m-1} nb_{n,m}(y)\right)\mathrm{d}y \leqslant K \sum_{m=1}^{\infty} \int\limits_{\mathbb{R}^3} mp_m(y)\,\mathrm{d}y < \infty.$$

By the dominated convergence theorem [4] and using (4),

$$\lim_{N,N_1\to\infty} \sum_{m=1}^{N} \int\limits_{B(O,N_1)} a_m(y)p_m(y)\left(\sum_{n=1}^{m-1} \int\limits_{|x|>N_1} b_{n,m}(y)h(x,n,m,y)n\,\mathrm{d}x\right)\mathrm{d}y$$

$$= \sum_{m=1}^{\infty} \int\limits_{\mathbb{R}^3} \sum_{n=1}^{m-1} na_m(y)p_m(y)b_{n,m}(y)\left(1 - \lim_{N_1\to\infty} \int\limits_{B(O,N_1)} h(x,n,m,y)\,\mathrm{d}x\right)\mathrm{d}y = 0.$$

Therefore

$$\sum_{m=1}^{\infty} \int\limits_{\mathbb{R}^3} \left(-a(x,n)p_n(x) + [\mathcal{B}_n p_n](x)\right)n\,\mathrm{d}x \geqslant 0,$$

which concludes the proof. □

5. Concluding remarks and discussion

The process of fragmentation with rate becoming infinite at infinity has been investigated by means of the theory of substochastic semigroups with a parameter and parameter-dependent operators. We succeeded to combine a discrete model with a nonlocal multiple fragmentation process with fragmentation rate depending on size and position and where new particles are spatially randomly distributed according to a given probabilistic law. We used Kato's Theorem and the dominated convergence theorem to get our main results here, that are conditions (21) and (22) that guarantee existence and conservativeness for the nonlocal discrete model described above and where each fragmentation rate $a_n(x)$ becomes infinite as $|x|$ is close to infinity. The physical interpretation is that the process is conservative if at infinity daughter particles tend to move back into the system with a high probability described by (22).

Acknowledgements

The research was partly made during the first edition of the Southern African Young Scientists Summer Program (SA-YSSP) hosted by the University of the Free State (UFS), South Africa, in collaboration with the International Institute for Applied Systems Analysis (IIASA) and supported by the National Research Foundation of South Africa (NRF) and the Department of Science and Technology (DST). Thanks to Prof. J. Banasiak for his inspiration.

References

[1] J. Banasiak, Kinetic-type models with diffusion: Conservative and nonconservative solutions, Transp. Theory Stat. Phys. 36 (1) (2007) 43–65.
[2] J. Banasiak, L. Arlotti, Perturbations of Positive Semigroups with Applications, Springer Monographs in Mathematics, 2006.
[3] J. Banasiak, W. Lamb, The discrete fragmentation equation: semigroups, compactness and asynchronous exponential growth, Kinet. Relat. Models 5 (2) (June 2012).
[4] R.G. Bartle, The Elements of Integration and Lebesgue Measure, 1st edition, Wiley–Interscience Publisher, 1995.
[5] C.R. Garibotti, G. Spiga, Boltzmann equation for inelastic scattering, J. Phys. A 27 (1994) 2709–2717.
[6] B. Haas, Loss of mass in deterministic and random fragmentation, Stochastic Process. Appl. 106 (2) (August 2003) 245–277.
[7] A. Majorana, C. Milazzo, Space homogeneous solutions of the linear semiconductor Boltzmann equation, J. Math. Anal. Appl. 259 (2) (2001) 609–629.
[8] S.C. Oukouomi Noutchie, E.F. Doungmo Goufo, On the honesty in nonlocal and discrete fragmentation dynamics in size and random position, ISRN Math. Anal. 2013 (2013), Article ID 908753, 7 p., http://dx.doi.org/10.1155/2013/908753.
[9] W. Wagner, Explosion phenomena in stochastic coagulation-fragmentation models, Ann. Appl. Probab. 15 (3) (2005) 2081–2112.
[10] R.M. Ziff, E.D. McGrady, The kinetics of cluster fragmentation and depolymerization, J. Phys. A 18 (1985) 3027–3037.
[11] R.M. Ziff, E.D. McGrady, "Shattering" transition in fragmentation, Phys. Rev. Lett. 58 (9) (1987) 892–895.