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Semicontinuous viscosity solutions for quasiconvex Hamiltonians [☆]



Solutions de viscosité semicontinues des hamiltoniens quasi-convexes

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ABSTRACT

The main theorem connecting convex Hamiltonians and semicontinuous viscosity solutions due to Barron and Jensen is extended to quasiconvex Hamiltonians. Some applications are indicated.

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R É S U M É

Le théorème principal reliant les hamiltoniens convexes et les solutions de viscosité semicontinues, due à Barron et Jensen, est étendu aux hamiltoniens quasi-convexes. Quelques applications sont indiquées.

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1. Introduction

In 1990, an extension of the Crandall–Lions notion of viscosity solutions for first order Hamilton–Jacobi equations was presented in [4]. This extension showed that when the Hamiltonian is convex in the gradient variable, touching the function from below and testing to see if the result gives zero, i.e., $p \in D^-u(x) \implies H(x, u(x), p) = 0$ is equivalent to the Crandall–Lions definition. This was important because, for lower semicontinuous functions, we may only be able to touch from below, and many problems only have semicontinuous solutions. The comparison principle for semicontinuous solutions was established in [4]. In 1993, H. Frankowska developed in [8] a nonsmooth approach to the uniqueness of semicontinuous viscosity solutions.

The aim of this note is to show that the idea of the connection between convex Hamiltonians and semicontinuous solutions extends to quasiconvex (equivalently, level convex) Hamiltonians, a much broader class of functions. Essentially all of the convex Hamiltonian results may be extended to quasiconvex Hamiltonians, except for one significant class, namely equations of the form $u_t + H(t, x, u, Du) = 0$. Assuming such an equation is quasiconvex in (u_t, Du) will force H to be convex in Du . Thus, finite-horizon Bolza or Lagrange problems are not covered. On the other hand, equations of the form $\lambda u + H(x, u, Du) = 0$ and time-dependent equations arising in optimal stopping and L^∞ control and calculus of variations problems are covered. Some applications for L^∞ problems are presented, as well as a representation result for a viscosity solution of $H(x, u, Du) = 0$.

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2. Semicontinuous solutions for quasiconvex Hamiltonians

The problem we consider is:

$$H(x, u(x), Du(x)) = 0, \quad x \in \Omega \subset \mathbb{R}^n. \quad (2.1)$$

In this paper, we will focus on Hamiltonians that are quasiconvex (also called level convex) in p :

$$H(x, r, \lambda p + (1 - \lambda)q) \leq \max\{H(x, r, p), H(x, r, q)\}, \quad 0 \leq \lambda \leq 1. \quad (2.2)$$

Equivalently, $E_\alpha(x, r) = \{p \in \mathbb{R}^n \mid H(x, r, p) \leq \alpha\}$ is convex for all $\alpha \in \mathbb{R}$. One of the fundamental features of quasiconvex functions is the extended Jensen inequality.

Proposition 2.1. (See [4,6].) Let $\mathcal{A} \subset \mathbb{R}^n$ be convex and $f : \mathcal{A} \rightarrow \mathbb{R}$ be quasiconvex and μ be a probability measure on \mathcal{A} . Then:

$$f\left(\int \varphi(x) d\mu(x)\right) \leq \mu - \operatorname{ess\,sup}_{x \in \mathcal{A}} f(\varphi(x)) \quad (2.3)$$

for any $\varphi \in L^1(\mu)$.

Proof. Consider $E_\alpha = \{z \in \mathcal{A} \mid f(z) \leq \alpha = \mu - \operatorname{ess\,sup}_{x \in \mathcal{A}} f(\varphi(x))\}$. Since f is quasiconvex, E_α is convex and $\varphi(x) \in E_\alpha$ for μ - almost all $x \in \mathcal{A}$. But then $\int \varphi(x) d\mu(x) \in E_\alpha$. \square

Definition 2.2 (Crandall–Lions). A bounded function u is a viscosity subsolution of (2.1) if $p \in D^+u^{usc}(x)$ implies $H_{lsc}(x, u, p) \leq 0$, and is a viscosity supersolution of (2.1) if $p \in D^-u_{lsc}(x)$ implies $H^{usc}(x, u_{lsc}, p) \geq 0$. Here a subscript lsc refers to the lower semicontinuous envelope of the function, and a superscript usc denotes the upper semicontinuous envelope.

Our goal in this section is to prove the following theorem. This extends one of the main theorems in [4] to quasiconvex Hamiltonians.

Theorem 2.3. Let $H(x, r, p)$ be continuous and satisfy $p \mapsto H(x, r, p)$ is quasiconvex and

$$|H(x, r, p) - H(x, s, p)| \leq \omega(|x - y|(|p| + 1) + |r - s|), \quad x \in \Omega \subset \mathbb{R}^n, r \in \mathbb{R}, p \in \mathbb{R}^n, \quad (2.4)$$

for $\omega \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\omega(0) = 0$. Then a uniformly continuous function $u : \Omega \rightarrow \mathbb{R}$ is a Crandall–Lions solution of $H(x, u(x), Du(x)) = 0$, $x \in \Omega$, if and only if

$$x \in \Omega, p \in D^-u(x) \implies H(x, u(x), p) = 0. \quad (2.5)$$

For brevity and in view of the literature since the introduction of such semicontinuous solutions in [4], we will refer to u as a BJ solution if it satisfies the condition (2.5).

There are two steps to prove this theorem. The first one assumes the stronger condition that u is Lipschitz.

Proposition 2.4. Let $u \in Lip(\Omega)$ and assume $p \mapsto H(x, r, p)$ is quasiconvex. Assume (see Ishii [9]) the condition:

$$H \text{ is continuous and } \forall R > 0, \limsup_{\varepsilon \rightarrow 0} \{ |H(y, s, p) - H(x, r, p)| \mid |y - x| < \varepsilon, |r - s| < \varepsilon, |p| \leq R \} = 0. \quad (2.6)$$

Then u is a Crandall–Lions viscosity subsolution of $H(x, u, Du) \leq 0$ if and only if u satisfies the condition $p \in D^-u(x) \implies H(x, u(x), p) \leq 0$, i.e., u is a BJ subsolution.

Proof. We use Ishii's [9] improved formulation of Theorem 1.1 in [4] that if u is bounded and lower semicontinuous (respectively, upper semicontinuous), then:

$$D^+u(x) \subset \bigcap_{r>0} \overline{co} \left(\bigcup_{|x-y|<r} D^-u(y) \right), \quad \text{respectively,} \quad D^-u(x) \subset \bigcap_{r>0} \overline{co} \left(\bigcup_{|x-y|<r} D^+u(y) \right). \quad (2.7)$$

Assume u is a Crandall–Lions subsolution. By (2.7), if $p \in D^-u(x)$, there is sequence $p_k \rightarrow p$, and $\{x_i\}$, $\{\lambda_i\}$, $\{q_i\}$, $i = 1, 2, \dots, N(k)$, $\exists N(k) \in \mathbb{Z}^+$, such that $|x - x_i| < \frac{1}{k}$, $q_i \in D^+u(x_i)$, $0 \leq \lambda_i \leq 1$, and $p_k = \sum \lambda_i q_i$, $\sum \lambda_i = 1$. Thus $H(x_i, u(x_i), q_i) \leq 0$. Observe that $|q_i| \leq Lip(u)$. Then, using (2.6), one shows $H(x, u(x), q_i) \leq o(\frac{1}{k}) \rightarrow 0$ as $k \rightarrow \infty$, $1 \leq i \leq N$. By quasiconvexity,

$$H(x, u(x), p_k) = H\left(x, u(x), \sum_{i=1}^{N(k)} \lambda_i q_i\right) \leq \max_{1 \leq i \leq N(k)} H(x, u(x), q_i) \leq o\left(\frac{1}{k}\right).$$

Sending $k \rightarrow \infty$ we get $H(x, u(x), p) \leq 0$. The converse can be shown in a similar way using the first part of (2.7). \square

Remark 2.5. An insight of Barles [3, Theorem 5.12] for the convex case makes transparent in the original convex case, and now in the quasiconvex case, why the theorem is true without requiring knowledge of (2.7). If u is Lipschitz and $H(x, u(x), Du(x)) \leq 0$ for almost every $x \in \Omega$, then if H is quasiconvex in p , it must be the case that u is a viscosity supersolution of $-H(x, u, Du) = 0$ in Ω . Here is how his idea works.

Let ρ be a $C^\infty(\mathbb{R}^n)$ regularizing kernel with $\rho \geq 0$, $\int_{\mathbb{R}^n} \rho \, dx = 1$. Setting $\rho_\varepsilon(x) = \varepsilon^{-n} \rho(x/\varepsilon)$ and $u_\varepsilon = u * \rho_\varepsilon$, we have:

$$\begin{aligned} H(x, u(x), Du_\varepsilon(x)) &\leq H(x, u_\varepsilon(x), Du_\varepsilon(x)) + o(1) \\ &= H\left(x, u_\varepsilon(x), \int_{\Omega'} Du(y) \rho_\varepsilon(x - y) \, dy\right) + o(1) \\ &\leq \mu - \text{ess sup}_{y \in \Omega'} H(y, u(y), Du(y)) + o(1) \\ &\leq o(1) \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0, \end{aligned}$$

in $\Omega' \Subset \Omega$. Here one uses the condition (2.6), $|Du| \leq R$, Proposition 2.1 and the fact that $\mu(A) = \int_A \rho_\varepsilon(x) \, dx$ is a probability measure, absolutely continuous with respect to Lebesgue measure.

Consequently, u_ε is a classical subsolution of $H(x, u_\varepsilon, Du_\varepsilon) - o(1) = 0$ and hence a classical supersolution of $-H(x, u_\varepsilon, Du_\varepsilon) + o(1) \geq 0$. By the Barles–Perthame stability results, $u = \lim_{\varepsilon \rightarrow 0}^* u_\varepsilon$ is a viscosity supersolution of $-H(x, u, Du) \geq 0$ in Ω . Observe how this argument would fail if we had an equation of the form $u_t + H(Du) = 0$ with $p \mapsto H(p)$ quasiconvex.

Using inf and sup convolutions, the proof of Theorem 2.3 now follows the same proof as the details in [2, Theorem 5.6] or [3, Theorem 2.2], with minor modifications.

Remark 2.6. As noted earlier, Theorem 2.3 does not apply to Hamilton–Jacobi equations of the form $u_t + H(x, u, Du) = 0$. The reason is the following lemma:

Lemma 2.7. Consider the function $f(x, y) = g(x) + h(y)$ where h is linear. The function f is quasiconvex in (x, y) if and only if g is convex.

Proof. This is Lemma 5.1 in [7]. \square

As a result of this lemma, if one assumes $(p_t, p_x) \rightarrow p_t + H(t, x, p_x)$ is quasiconvex, we are back in the case when the Hamiltonian is assumed convex and there is nothing new. On the other hand, the theorem does apply to stationary Hamilton–Jacobi equations $u + H(x, u, Du) = 0$ and Hamilton–Jacobi equations involving optimal stopping or L^∞ control problems (see below).

We state the following theorem whose proof, given Theorem 2.3, is a minor modification to that in Barles [3, Theorem 5.14] and Bardì and Capuzzo–Dolcetta [2, Theorem 5.5]. First we give Soravia’s [10] definition of a BJ solution for a Dirichlet problem.

Definition 2.8. Let g be lower semicontinuous (lsc) on \mathbb{R}^n . The lsc function u is a BJ solution of $u + H(x, u, Du) = 0$, $x \in \Omega$, $u = g$, $x \in \partial\Omega$, if $x \in \Omega$, $p \in D^-u(x) \implies u(x) + H(x, u(x), p) \geq 0$, $u = g$, for $x \in \Omega^c$, and $x \in \mathbb{R}^n$, $p \in D^-u(x) \implies -u - H(x, u(x), p) \geq 0$.

Theorem 2.9. Assume the conditions that for constants $C_H^x, C_H^r, C_H^p > 0$, $r \mapsto H(t, x, r, p)$ is nondecreasing, $p \mapsto H(x, r, p)$ is quasiconvex, and:

$$\begin{aligned} |H(x, r, p) - H(y, r, p)| &\leq C_H^x(1 + |p|)|x - y|, & |H(x, r, p) - H(x, s, p)| &\leq C_H^r|r - s|, \\ |H(x, r, p) - H(x, r, q)| &\leq C_H^p(1 + |x|)|p - q|. \end{aligned}$$

Suppose that u is bounded lsc on \mathbb{R}^n , v is bounded lsc on $\bar{\Omega}$, with u satisfying $x \in \mathbb{R}^n$, $p \in D^-u(x) \implies u + H(x, u, p) \leq 0$ and v satisfying $x \in \Omega$, $p \in D^-v(x) \implies v + H(x, v, Dv) \geq 0$. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω .

Sketch of proof. This uses Barles' idea of modified convolutions. Let $u_\varepsilon(t, x) = \inf_{y \in \mathbb{R}^n} u(y) + e^{-Kt} \frac{|x-y|^2}{\varepsilon^2}$. Then u_ε is a Crandall–Lions subsolution of $u_{\varepsilon,t} + u_\varepsilon + H(x, u_\varepsilon, Du_\varepsilon) \leq C_H^x \varepsilon e^{Kt/2} m_0$, $(t, x) \in \mathbb{R}^{n+1}$, where m_0 is a constant depending only on the supremum bound of u . Indeed, let $y_\varepsilon \in \arg \min(u(y) + e^{-Kt} \frac{|x-y|^2}{\varepsilon^2})$. Then $u(y_\varepsilon) + H(y_\varepsilon, u(y_\varepsilon), p_\varepsilon) \leq 0$, $p_\varepsilon = 2e^{-Kt}(x - y_\varepsilon)/\varepsilon^2$, and $|x - y_\varepsilon| \leq \varepsilon e^{Kt/2} m_0$. Then

$$\begin{aligned} u_{\varepsilon,t} + u_\varepsilon + H(x, u_\varepsilon, Du_\varepsilon) &\leq e^{-Kt} \frac{|x - y_\varepsilon|^2}{\varepsilon^2} (-K + 2C_H^x + C_h^f + 1) + C_H^x |x - y_\varepsilon| \\ &\leq C_H^x \varepsilon e^{Kt/2} m_0, \quad \text{choosing } K = 2C_H^x + 1 + C_H^u. \end{aligned}$$

The rest of the proof is almost exactly the same as [2, Theorem 5.5, p. 346]. \square

2.1. Applications

Quasiconvex Hamiltonians occur in many physical problems, but here we sketch the main application arising in control theory of the following form:

$$H(x, r, p) = \max_{z \in Z} \min \{ \lambda r - f(x, z) \cdot p - k(x, z), r - h(x, z) \}, \quad x \in \Omega \subset \mathbb{R}^n, \lambda > 0 \quad (2.8)$$

or, for $(t, x) \in (0, \infty) \times \mathbb{R}^n$

$$H(t, x, r, (p_t, p_x)) = \max_{z \in Z} \min \{ p_t - f(t, x, z) \cdot p_x - k(t, x, z), r - h(t, x, z) \}. \quad (2.9)$$

Proposition 2.10. *The Hamiltonians in (2.8)–(2.9) are quasiconvex in the p variable.*

Proof. We will show that (2.9) is quasiconvex in $p = (p_t, p_x)$. Since (t, x, r) are fixed, it is sufficient to show $\eta(p) = \max_{z \in Z} (p_t - p_x \cdot y(z) - w(z)) \wedge \gamma(z)$ is quasiconvex. Fix $z \in Z$. Let $\beta \in \mathbb{R}$. If $\gamma(z) \leq \beta$ then $E_\beta = \{p = (p_t, p_x) \in \mathbb{R}^{n+1} \mid (p_t - p_x \cdot y(z) - w(z)) \wedge \gamma(z) \leq \beta\} = \mathbb{R}^{n+1}$. If $\gamma(z) > \beta$, $E_\beta = \{p_t - p_x \cdot y(z) - w(z) \leq \beta\}$ is a halfspace in \mathbb{R}^{n+1} and so convex. Since the maximum of a collection of quasiconvex functions is quasiconvex, $\eta(p)$ is quasiconvex. \square

The solution of $H(x, u, Du) = 0$ in (2.8) or $H(t, x, u, u_t, Du) = 0$ in (2.9) is given as the value function of an optimal control problem in L^∞ . For example, the solution of $H(t, x, u, u_t, Du) = 0$ in $(0, \infty) \times \mathbb{R}^n$, $u(0, x) = g(x) \vee \min_z h(0, x, z)$ is given by:

$$u(t, x) = \inf_{\zeta \in \mathcal{Z}[0, t]} \operatorname{ess\,sup}_{s \in [0, t]} \left(h(s, \xi(s), \zeta(s)) + \int_0^s k(\tau, \xi(\tau), \zeta(\tau)) \, d\tau \right) \vee \left(g(\xi(t)) + \int_0^t k(\tau, \xi(\tau), \zeta(\tau)) \, d\tau \right).$$

Here $\dot{\xi} = f(\tau, \xi, \zeta)$, $\xi(0) = x$ and $\mathcal{Z}[0, t]$ is the class of control functions $\zeta : [0, t] \rightarrow Z$. The relaxed version, i.e., with $z \mapsto h(t, x, z)$ quasiconvex and $z \mapsto k(t, x, z)$ convex result in a value function u which is the unique lower semicontinuous B] solution of $H(t, x, u, u_t, Du) = 0$ satisfying the initial condition. Details for the case $k = 0$ can be found in [6].

Remark 2.11. It is reasonable to ask if we may represent the solution of general Hamilton–Jacobi equations as the value function of such a control problem. This can be worked out for a wide class of equations. We recall the relevant notion of quasiconvex conjugate introduced in Barron and Liu [6]. Assume a function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ satisfies the condition: For every $\gamma < \sup f$, there is a continuous affine function which is a minorant of f on $\{f(x) \leq \gamma\}$. Given such a function, define the symmetric conjugates:

$$\begin{aligned} f^\% (r, p) &= \sup_{x \in \mathbb{R}^n} (p \cdot x \wedge r - f(x)) = \sup_{x \in \mathbb{R}^n} (p \cdot x - f(x)) \wedge (r - f(x)), \\ f^{\%\%} (x) &= \sup_{r \in \mathbb{R}, p \in \mathbb{R}^n} (p \cdot x \wedge r - f^\% (r, p)). \end{aligned} \quad (2.10)$$

It is proved in [1,5] that a lower semicontinuous function f satisfying the condition is quasiconvex if and only if $f = f^{\%\%}$.

We give an example of a result possible for the stationary case. We are given $H : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ with $p \mapsto H(x, r, p)$ quasiconvex and $r \mapsto H(x, r, p)$ nondecreasing. Define $H^\%(x, z) = \sup_{r, p} \{(p \cdot z \wedge r - H(x, r, p))\}$. Then motivated by the quasiconvex conjugates (2.10), we have $H(x, r, p) = \sup_{z \in \mathbb{R}^n} (p \cdot z - H^\%(x, z)) \wedge (r - H^\%(x, z))$. Then $z \mapsto H^\%(x, z)$ is quasiconvex. With this representation for H , we have $H(x, u, Du) = \sup_{z \in \mathbb{R}^n} (z \cdot Du - H^\%(x, z)) \wedge (u - H^\%(x, z))$. Now we may represent a viscosity solution of $H(x, u, Du) = 0$ in \mathbb{R}^n as:

$$u(x) = \inf_{\xi(\cdot), \xi(0)=x} \operatorname{ess\,sup}_{s \in [0, \infty)} \left(H^\% (\xi(s), -\dot{\xi}(s)) + \int_0^s H^\% (\xi(\tau), -\dot{\xi}(\tau)) \, d\tau \right).$$

This is similar to the classic Lax–Oleinik formula, but our formula includes u dependence which does not arise as simply a discount factor. Similar representation formulas are possible for the Dirichlet problem. Variations of the conjugates given in [1] give formulas for time-dependent problems with convex Hamiltonians and u dependence. This will be considered elsewhere.

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