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Harmonic Analysis

Measures with uniformly discrete support and spectrum ☆

Mesures à support et spectre uniformément discrets

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ABSTRACT

We characterize the measures on $\mathbb{R}$ which have both their support and spectrum uniformly discrete. A similar result is obtained in $\mathbb{R}^n$ under a stronger discreteness restriction.

RÉSUMÉ

Nous caractérisons les mesures sur $\mathbb{R}$ ayant toutes les deux leurs support et spectre uniformément discrets. Un résultat similaire est obtenu dans $\mathbb{R}^n$ sous une restriction de discrétion plus forte.

1. Introduction. Results

1.1. A set $\Lambda \subset \mathbb{R}^n$ is called uniformly discrete (u.d.) if

$$\mathrm{d}(\Lambda) := \inf_{\lambda,\lambda' \in \Lambda, \lambda \neq \lambda'} |\lambda - \lambda'| > 0.$$

We consider a measure μ on $\mathbb{R}^n$ supported on a u.d. set Λ:

$$\mu = \sum_{\lambda \in \Lambda} \mu(\lambda)\delta_\lambda, \quad \mu(\lambda) \neq 0,\ \mathrm{d}(\Lambda) > 0. \tag{1}$$

Assume that μ is a temperate distribution, and that its Fourier transform

$$\hat{\mu}(x) := \sum_{\lambda \in \Lambda} \mu(\lambda) e^{-2\pi i \langle \lambda, x \rangle}$$

(in the sense of distributions) is also a measure, supported by a u.d. set S:

$$\hat{\mu} = \sum_{s \in S} \hat{\mu}(s)\delta_s, \quad \hat{\mu}(s) \neq 0,\ \mathrm{d}(S) > 0. \tag{2}$$

The set S is the spectrum of the measure μ.

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The classical Poisson summation formula provides an example of such a situation:

$$\mu = \sum_{k\in\mathbb{Z}^n} \delta_k. \tag{3}$$

In this case $\hat{\mu} = \mu$.

Kahane and Mandelbrojt [4] studied the problem (in one dimension), which other summation formulas of Poisson type may exist.

There is a conjecture (see, e.g., [7]) that (3) is essentially the only possible example of a measure μ satisfying (1) and (2). Namely, that the support of such a measure is contained in a finite union of translates of a (full-rank) lattice.

Under the assumption that all the masses $\mu(\lambda)$ are equal, or take only finitely many different values, such results were proved in [8, p. 25], [2], [5]. The proofs are based on the Cohen–Helson theorem on idempotent measures. See also [1].

The aim of the present note is to sketch a proof of the conjecture above. For $n = 1$ it is obtained in all generality, while for $n > 1$ under a stronger "quasi-regularity" condition on the spectrum.

Theorem 1. *Let μ be a measure on $\mathbb{R}$ satisfying (1) and (2). Then the support Λ is contained in a finite union of translates of a certain lattice. The same is true for S (with the dual lattice).*

Theorem 2. *Let μ be a measure on $\mathbb{R}^n$, $n > 1$, satisfying (1) and (2), and such that $S - S$ is a u.d. set. Then the conclusion of Theorem 1 holds.*

The following proposition completes the results, describing the explicit form of μ.

Theorem 3. *Let μ be a measure on $\mathbb{R}^n$, $n \geqslant 1$, satisfying (1) and (2), and such that Λ is contained in a finite union of translates of a lattice L. Then μ is of the form*

$$\mu = \sum_{j=1}^{N} c_j \sum_{\ell\in L} \mathrm{e}^{\mathrm{i}\langle\theta_j,\ell\rangle}\, \delta_{\ell+\omega_j} \tag{4}$$

where ω_j, θ_j are vectors in $\mathbb{R}^n$, and c_j are complex numbers ($1 \leqslant j \leqslant N$).

Conversely, every measure μ of the form (4) satisfies (1) and (2).

2. Proof of Theorem 1

Here we sketch the proof of Theorem 1. We consider a measure μ on $\mathbb{R}^n$ satisfying (1) and (2). Only in Section 2.3 the specifics of the one-dimensional case are used.

2.1. We will use the following notation: for $h \in \Lambda - \Lambda$, denote

$$\Lambda_h := \Lambda \cap (\Lambda - h) = \{\lambda \in \Lambda:\ \lambda + h \in \Lambda\}.$$

Lemma 1. *For every $h \in \Lambda - \Lambda$ and $r > 0$, there is a non-zero finite measure ν_h supported by Λ_h, whose spectrum lies in the r-neighborhood of the set $S - S$.*

Proof. Multiply μ by a function $\varphi > 0$ in the Schwartz class, whose spectrum lies in the ball $B_r := \{x \in \mathbb{R}^n:\ |x| < r\}$. Since μ is a temperate distribution, this yields another measure

$$\mu_1 = \sum_{\lambda\in\Lambda} c(\lambda)\delta_\lambda$$

such that

(i) $c(\lambda) \neq 0$, $\sum |c(\lambda)| < \infty$;
(ii) $\operatorname{spec}(\mu_1) \subset S + B_r$.

Consider the Fourier transform of μ_1:

$$f(x) := \sum_{\lambda\in\Lambda} c(\lambda)\mathrm{e}^{-2\pi \mathrm{i}\langle\lambda,x\rangle}.$$

Clearly, f is a bounded, continuous function on $\mathbb{R}^n$, and vanishes outside of $S + B_r$.

For $u \in \mathbb{R}^n$, set:

$$g(x,u) := f(x+u)\overline{f(x)} = \sum_{\lambda\in\Lambda}\sum_{\lambda'\in\Lambda} c(\lambda)\overline{c(\lambda')}\mathrm{e}^{2\pi\mathrm{i}(\langle\lambda'-\lambda,x\rangle-\langle\lambda,u\rangle)}$$

$$= \sum_{h\in\Lambda-\Lambda} \mathrm{e}^{2\pi\mathrm{i}\langle h,x\rangle}\left[\sum_{\lambda\in\Lambda_h} c(\lambda)\overline{c(\lambda+h)}\mathrm{e}^{-2\pi\mathrm{i}\langle\lambda,u\rangle}\right].$$

Denote the quantity in brackets by $A_h(u)$. Clearly

$$g(x,u) = \sum_{h\in\Lambda-\Lambda} A_h(u)\mathrm{e}^{2\pi\mathrm{i}\langle h,x\rangle}$$

vanishes identically (with respect to x) for each

$$u \in U := \mathbb{R}^n \setminus \big[(S-S)+B_{2r}\big].$$

It follows that $A_h(u) = 0$ $(u \in U)$. Consider the measure

$$\nu_h := \sum_{\lambda\in\Lambda_h} c(\lambda)\overline{c(\lambda+h)}\delta_\lambda,$$

then we have $\hat{\nu}_h = A_h$. It follows that $\hat{\nu}_h(u) = 0$ $(u \in U)$. □

2.2. Notice that if r is sufficiently small, then the set U above contains a ball centered near zero, of radius $a = a(S) > 0$. So ν_h is a finite non-zero measure with a spectral gap of radius a.

Corollary. *The set Λ (and S) in (1), (2) cannot be rationally independent.*

We refer to [4] where several conclusions concerning the arithmetical structure of Λ, S are obtained in the one-dimensional setting.

2.3. In a recent paper [11], a characterization is given of u.d. sets in $\mathbb{R}$ that may support a finite measure with a spectral gap of given size, in terms of the lower Beurling–Malliavin density.

For our goal a simple necessary condition is enough, which admits an independent proof, similar to the one used in [13, pp. 1044–1045].

Lemma 2. *If a u.d. set $\Lambda \subset \mathbb{R}$ supports a measure with a spectral gap of size $a > 0$, then*

$$D_\#(\Lambda) := \liminf_{R\to\infty} \frac{\#(\Lambda\cap B_R)}{|B_R|} > c\big(a, d(\Lambda)\big) > 0. \tag{5}$$

2.4.

Lemma 3. *If $\Lambda \subset \mathbb{R}^n$ is a u.d. set such that $D_\#(\Lambda_h) > c(\Lambda) > 0$ $(h \in \Lambda - \Lambda)$, then*

$$D^+(\Lambda-\Lambda) < \infty. \tag{6}$$

Here

$$D^+(\Lambda) := \limsup_{R\to\infty}\ \sup_{x\in\mathbb{R}^n} \frac{\#(\Lambda\cap(x+B_R))}{|B_R|}$$

is the Kahane–Beurling upper uniform density.

2.5. Now we need the concepts of Delone and Meyer sets in $\mathbb{R}^n$.

Definition. Λ is called a Delone set if Λ is u.d. and relatively dense.

Definition. Λ is called a Meyer set if the following two conditions are satisfied:

(i) Λ is a Delone set;
(ii) There is a finite set F such that $\Lambda - \Lambda \subset \Lambda + F$.

Lagarias [6] proved that if Λ is a Delone set and $\Lambda - \Lambda$ is u.d., then Λ is a Meyer set (see also [12]). Using a similar argument one can prove:

Lemma 4. *If Λ is a Delone set such that $D^+(\Lambda - \Lambda) < \infty$, then Λ is a Meyer set.*

2.6.

Lemma 5. *If Λ is a Meyer set and if*

$$D^+(\Lambda_h) > c(\Lambda) > 0 \quad (h \in \Lambda - \Lambda) \tag{7}$$

then Λ is contained in a finite union of translates of a lattice.

Proof. By a theorem of Meyer [9, Sections II.5, II.14] (see also [12]) we have $\Lambda \subset M + F$, where F is a finite set and M is a "model set". The latter means that there is a lattice $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^m$ ($m \geqslant 0$) and a bounded set $\Omega \subset \mathbb{R}^m$ such that

$$M = M\big(\mathbb{R}^n \times \mathbb{R}^m, \Gamma, \Omega\big) := \big\{p_1(\gamma):\ \gamma \in \Gamma, p_2(\gamma) \in \Omega\big\}, \tag{8}$$

where $p_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $p_2 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ are the canonical projections, p_1 restricted to Γ is injective, and $p_2(\Gamma)$ is dense in $\mathbb{R}^m$.

Thus any $\lambda \in \Lambda$ admits a representation

$$\lambda = p_1(\gamma_\lambda) + u_\lambda, \quad \gamma_\lambda \in \Gamma,\ p_2(\gamma_\lambda) \in \Omega,\ u_\lambda \in F.$$

Let $E := \{p_2(\gamma_\lambda):\ \lambda \in \Lambda\}$. Given $\delta > 0$, choose $\lambda_0, \lambda_0' \in \Lambda$ such that

$$\big|p_2(\gamma_{\lambda_0'}) - p_2(\gamma_{\lambda_0})\big|^2 > (\operatorname{diam} E)^2 - \delta^2,$$

and set $h := \lambda_0' - \lambda_0$. Then $h \in \Lambda - \Lambda$.

One can prove that M and F may be chosen such that $p_1(\Gamma) \cap (F + F - F - F) = \{0\}$. It follows that if $\lambda \in \Lambda_h$ then

$$p_2(\gamma_{\lambda+h}) - p_2(\gamma_\lambda) = p_2(\gamma_{\lambda_0+h}) - p_2(\gamma_{\lambda_0}),$$

which in turn implies

$$p_2(\gamma_\lambda) \in \Omega' := p_2(\gamma_{\lambda_0}) + B_\delta.$$

This shows that

$$\Lambda_h \subset M' + F, \quad M' = M'\big(\mathbb{R}^n \times \mathbb{R}^m, \Gamma, \Omega'\big).$$

Now suppose that $m \geqslant 1$. Since $D^+(M') = (\det \Gamma)^{-1}|\Omega'|$, it follows that $D^+(\Lambda_h) < \varepsilon$ if δ is sufficiently small, which is in contradiction with (7). Hence $m = 0$, and M must be a lattice. □

2.7. Theorem 1 now follows. Indeed, (2) implies that Λ is a Delone set (see Lemma 1 in [2]). This together with (5) gives (6) (Lemma 3). So Λ is a Meyer set (Lemma 4). Lemmas 1 and 2 imply (7). Now Lemma 5 finalizes the proof.

3. Proof of Theorem 2

Now we sketch the proof of Theorem 2.

Lemma 6. *Given a number $a > 0$ there is $R = R(n, a)$ such that, if a measure ν is supported by a u.d. set Q in $\mathbb{R}^n$, $\mathrm{d}(Q) > a$, and if $\hat{\nu}$ vanishes on a ball B_R, then $\nu = 0$.*

This lemma follows from Ingham type theorems used in the interpolation theory in $\mathbb{R}^n$ (see for example [14]).

Lemma 6 allows one to avoid using the one-dimensional Lemma 2 and gives that Λ_h is a Delone set with uniform estimate: every ball of radius R (independent of h) intersects Λ_h. In turn, this implies (6) and (7). Now the proof of Theorem 2 can be finished as above.

We skip the proof of Theorem 3.

4. Remarks

It should be mentioned that if one requires S to be just a countable set, then the result fails. As an example, one may take Meyer's quasicrystals, namely the model set M defined by (8) (with $m \geqslant 1$). Then M is a u.d. set, which supports a measure μ whose Fourier transform is a sum of point masses (see [10]), but M is not contained in a finite union of translates of a lattice.

See also [3] where possible applications of general quasicrystals are discussed.

Note added in proof

At present we have proved Theorem 2 for positive measures μ in $\mathbb{R}^n$, without the assumption that $S - S$ is u.d. The proof will be published elsewhere.

References

[1] A. Córdoba, La formule sommatoire de Poisson, C. R. Acad. Sci. Paris, Ser. I 306 (1988) 373–376.
[2] A. Córdoba, Dirac combs, Lett. Math. Phys. 17 (1989) 191–196.
[3] F. Dyson, Birds and frogs, Not. Am. Math. Soc. 56 (2009) 212–223.
[4] J.-P. Kahane, S. Mandelbrojt, Sur l'équation fonctionnelle de Riemann et la formule sommatoire de Poisson, Ann. Sci. Éc. Norm. Super. 75 (1958) 57–80.
[5] M.N. Kolountzakis, J.C. Lagarias, Structure of tilings of the line by a function, Duke Math. J. 82 (1996) 653–678.
[6] J.C. Lagarias, Meyer's concept of quasicrystal and quasiregular sets, Commun. Math. Phys. 179 (1996) 365–376.
[7] J.C. Lagarias, Mathematical quasicrystals and the problem of diffraction, in: Directions in Mathematical Quasicrystals, in: CRM Monogr. Ser., vol. 13, Amer. Math. Soc., Providence, 2000, pp. 61–93.
[8] Y. Meyer, Nombres de Pisot, nombres de Salem et analyse harmonique, Lect. Notes Math., vol. 117, Springer-Verlag, 1970.
[9] Y. Meyer, Algebraic Numbers and Harmonic Analysis, N.-Holl. Math. Libr., vol. 2, North-Holland Publishing Co./American Elsevier Publishing Co., Inc., Amsterdam–London/New York, 1972.
[10] Y. Meyer, Quasicrystals, Diophantine approximation and algebraic numbers, in: Beyond Quasicrystals, Les Houches, 1994, Springer, Berlin, 1995, pp. 3–16.
[11] M. Mitkovski, A. Poltoratski, Pólya sequences, Toeplitz kernels and gap theorems, Adv. Math. 224 (2010) 1057–1070.
[12] R.V. Moody, Meyer sets and their duals, in: The Mathematics of Long-Range Aperiodic Order, Waterloo, ON, 1995, in: NATO Adv. Stud. Inst. Ser., Ser. C, Math. Phys. Sci., vol. 489, Kluwer Acad. Publ., Dordrecht, The Netherlands, 1997, pp. 403–441.
[13] A. Olevskii, A. Ulanovskii, Universal sampling and interpolation of band-limited signals, Geom. Funct. Anal. 18 (2008) 1029–1052.
[14] A. Olevskii, A. Ulanovskii, On multi-dimensional sampling and interpolation, Anal. Math. Phys. 2 (2012) 149–170.