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Functional Analysis

A Hilbert-type integral inequality with the mixed kernel of multi-parameters ☆

Une inégalité intégrale de type Hilbert avec noyau mixte dépendant de plusieurs paramètres

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ABSTRACT

In this paper, by means of the weight function and the technique of real analysis, and introducing the Γ-function and the Riemann ζ-function to jointly characterize the constant factor, a Hilbert-type integral inequality with the mixed kernel of multi-parameters and its equivalent form are given; their constant factors are proved to be the best possible. By selecting special parameter values, some meaningful results are obtained.

RÉSUMÉ

Dans ce texte, nous obtenons, sous deux formes équivalentes, une inégalité intégrale de type Hilbert, avec un noyau mixte dépendant de plusieurs paramètres. Nous utilisons à cette fin des fonctions poids, des techniques d'analyse réelle et les fonctions gamma d'Euler et zéta de Riemann, afin d'expliciter le facteur constant (c'est-à-dire ne dépendant que des paramètres), dont il est démontré qu'il est le meilleur possible. En choisissant des valeurs spéciales des paramètres, nous en déduisons quelques résultats significatifs.

1. Introduction

For convenience, if $\theta(x)(>0)$ is a measurable function, $\rho \geqslant 1$, the function spaces are set as:

$$L^{\rho}(0,\infty) := \left\{ h \geqslant 0;\ \|h\|_{\rho} := \left\{ \int_0^{\infty} h^{\rho}(x)\,\mathrm{d}x \right\}^{\frac{1}{\rho}} < \infty \right\}, \tag{1.1}$$

and

$$L^{\rho}_{\theta}(0,\infty) := \left\{ h \geqslant 0;\ \|h\|_{\rho,\theta} := \left\{ \int_0^{\infty} \theta(x) h^{\rho}(x)\,\mathrm{d}x \right\}^{\frac{1}{\rho}} < \infty \right\}. \tag{1.2}$$

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If $f, g \geqslant 0$, $f, g \in L^2(0,\infty)$, $0 < \|f\|_2 < \infty$, $0 < \|g\|_2 < \infty$, then we have ([1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,\mathrm{d}x\,\mathrm{d}y < \pi \|f\|_2 \|g\|_2, \tag{1.3}$$

where the constant factor π is the best possible. Inequality (1.3) is the famous Hilbert integral inequality, which is important in analysis and its applications [1,7]. In recent years, inequality (1.3) had been improved and extended by [5,9–11]. Recently, two Hilbert-type integral inequalities with the best constant factor were obtained [6,11] as:

$$\int_0^\infty \int_0^\infty \mathrm{e}^{-xy} f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y < \sqrt{\pi}\|f\|_2\|g\|_2, \tag{1.4}$$

$$\int_0^\infty \int_0^\infty \operatorname{sech}(xy) f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y < 2c_0\|f\|_{2,\varphi}\|g\|_{2,\psi}, \tag{1.5}$$

where $c_0 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 0.9159655942^+$ is Catalan's constant, $\varphi(x) = x^{-3}$, $\psi(y) = y^{-3}$.

In this paper, by means of the weight function and the technique of real analysis, a Hilbert-type integral inequality with mixed kernel is given as follows:

If $\varphi(x) = x^{-3}$, $\psi(y) = y^{-3}$, $f \in L^2_\varphi(0,\infty)$, $g \in L^2_\psi(0,\infty)$, $\|f\|_{2,\varphi}, \|g\|_{2,\psi} > 0$, then:

$$\int_0^\infty \int_0^\infty \mathrm{e}^{-xy} \tanh(xy) f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y < (2c_0 - 1)\|f\|_{2,\varphi}\|g\|_{2,\psi}, \tag{1.6}$$

where c_0 is Catalan's constant.

2. Some lemmas

Lemma 2.1. *Let $a > -1$, $\mathrm{Re}(s) > 0$, then the Laplace integral transform of the power function x^a as follows* ([8]):

$$F(s) = \int_0^\infty x^a \mathrm{e}^{-sx}\,\mathrm{d}x = \frac{\Gamma(a+1)}{s^{a+1}}, \tag{2.1}$$

where $\Gamma(z)$ is the Γ-function $\Gamma(z) = \int_0^\infty \mathrm{e}^{-u}u^{z-1}\,\mathrm{d}u$ $(z > 0)$.

Lemma 2.2. *Let m be a positive integer, then we have the summation formulas* ([2]):

$$S_1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2m-1}} = \frac{\pi^{2m+1}E_m}{2^{2m+2}(2m)!}, \tag{2.2}$$

where the E_m's are the Euler numbers, viz. $E_0 = 1$, $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, $E_5 = 50\,521$, etc., and

$$S_2 = \sum_{k=1}^{\infty} \frac{1}{k^{2m}} = \frac{2^{2m-1}\pi^{2m}}{(2m)!}B_m, \tag{2.3}$$

where the B_m's are the Bernoulli numbers, viz. $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, $B_5 = \frac{5}{66}$, $B_6 = \frac{691}{2730}$, $B_7 = \frac{7}{6}$, etc.

Lemma 2.3. *If $\beta > 0$, we have*:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^{\beta+1}} = \left(1 - \frac{1}{2^\beta}\right)\zeta(\beta+1), \tag{2.4}$$

where $\zeta(x)$ is Riemann's ζ-function ($\zeta(x) = \sum_{k=1}^{\infty} \frac{1}{k^x}$ $(x > 1)$).

Proof. Because

$$\zeta(\beta+1)=\sum_{k=1}^{\infty}\frac{1}{k^{\beta+1}}=\sum_{k=1}^{\infty}\frac{1}{(2k)^{\beta+1}}+\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{\beta+1}}=\frac{1}{2^{\beta+1}}\zeta(\beta+1)+\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{\beta+1}},$$

therefore

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{\beta+1}}=\left(1-\frac{1}{2^{\beta+1}}\right)\zeta(\beta+1), \tag{2.5}$$

and, by (2.5), we find:

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^{\beta+1}}=\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{\beta+1}}-\sum_{k=1}^{\infty}\frac{1}{(2k)^{\beta+1}}=\left(1-\frac{1}{2^{\beta}}\right)\zeta(\beta+1). \qquad \square$$

Lemma 2.4. *If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\alpha,\beta>0$, we define the weight function as*:

$$\omega(\alpha,\beta,x):=\int_0^{\infty}e^{-\alpha xy}\tanh(\beta xy)\frac{y^{\beta}}{x^{\frac{p\beta}{q}}}\,dy,\quad x\in(0,+\infty),$$

$$\omega(\alpha,\beta,y):=\int_0^{\infty}e^{-\alpha xy}\tanh(\beta xy)\frac{x^{\beta}}{y^{\frac{q\beta}{p}}}\,dx,\quad y\in(0,+\infty),$$

then we have:

$$\omega(\alpha,\beta,x)=C(\alpha,\beta)x^{-p\beta-1},\qquad \omega(\alpha,\beta,y)=C(\alpha,\beta)y^{-q\beta-1},$$

where

$$C(\alpha,\beta)=\frac{1}{2^{\beta}\beta^{\beta+1}}\Gamma(\beta+1)\left[\sum_{k=0}^{\infty}(-1)^k\frac{1}{(k+\frac{\alpha}{2\beta})^{\beta+1}}-\frac{1}{2}\left(\frac{2\beta}{\alpha}\right)^{\beta+1}\right]. \tag{2.6}$$

1^0 *When $\alpha=2\beta$, by (2.6) and (2.4), we find*:

$$C(\alpha,\beta)=\frac{\Gamma(\beta+1)}{2^{\beta}\beta^{\beta+1}}\left[\left(1-\frac{1}{2^{\beta}}\right)\zeta(\beta+1)-\frac{1}{2}\right]. \tag{2.7}$$

2^0 *When $\alpha=2\beta$, $\beta=2m-1$ $(m=1,2,\ldots)$, by (2.7) and (2.3), we find*:

$$C(\alpha,\beta)=\frac{(2^{2m-1}-1)\pi^{2m}B_m}{m2^{2m}(2m-1)^{2m}}-\frac{(2m-1)!}{2^{2m}(2m-1)^{2m}}, \tag{2.8}$$

where the B_m's are the Bernoulli numbers.

3^0 *When $\alpha=\beta=2m$ $(m=1,2,\ldots)$, by (2.6) and (2.2), we find*:

$$C(\alpha,\beta)=\frac{\pi^{2m+3}E_{m+1}}{(m+1)(2m+1)2^{2m+4}(2m)^{2m+1}}-\frac{(2m)!}{(2m)^{2m+1}}, \tag{2.9}$$

where the E_m's are the Euler numbers.

Proof. Setting $\beta xy=u$, then by (2.1), we have:

$$\begin{aligned}\omega(\alpha,\beta,x)&=\int_0^{\infty}e^{-\alpha xy}\tanh(\beta xy)\frac{y^{\beta}}{x^{\frac{p\beta}{q}}}\,dy=\frac{1}{\beta^{\beta+1}}x^{-p\beta-1}\int_0^{\infty}\frac{e^{-\frac{\alpha}{\beta}u}-e^{-(2+\frac{\alpha}{\beta})u}}{1+e^{-2u}}u^{\beta}\,du\\&=\frac{1}{\beta^{\beta+1}}x^{-p\beta-1}\sum_{k=0}^{\infty}(-1)^k\left\{\frac{\Gamma(\beta+1)}{(2k+\frac{\alpha}{\beta})^{\beta+1}}-\frac{\Gamma(\beta+1)}{[2(k+1)+\frac{\alpha}{\beta}]^{\beta+1}}\right\}\\&=\frac{1}{2^{\beta}\beta^{\beta+1}}x^{-p\beta-1}\Gamma(\beta+1)\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+\frac{\alpha}{2\beta})^{\beta+1}}-\frac{1}{2}\left(\frac{2\beta}{\alpha}\right)^{\beta+1}\right]=C(\alpha,\beta)x^{-p\beta-1}.\end{aligned}$$

By the same way, we obtain $\omega(\alpha,\beta,y)=C(\alpha,\beta)y^{-q\beta-1}$. $\square$

Lemma 2.5. *If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\alpha,\beta>0$, $0<\varepsilon<\min\{q\beta,p\beta\}$, and ε small enough, let us define the real functions as follows*:

$$\tilde{f}(x):=\begin{cases}0, & x\in(0,1),\\ x^{\frac{p\beta-\varepsilon}{p}}, & x\in[1,\infty),\end{cases}\qquad \tilde{g}(y):=\begin{cases}0, & y\in(1,\infty),\\ y^{\frac{q\beta+\varepsilon}{q}}, & y\in(0,1],\end{cases}$$

then we have:

$$\tilde{J}\varepsilon=\left[\int_0^\infty x^{-p\beta-1}\tilde{f}^p(x)\,\mathrm{d}x\right]^{\frac{1}{p}}\left[\int_0^\infty y^{-q\beta-1}\tilde{g}^q(y)\,\mathrm{d}y\right]^{\frac{1}{q}}\varepsilon=1, \tag{2.10}$$

$$\tilde{I}\varepsilon=\varepsilon\int_0^\infty\int_0^\infty \mathrm{e}^{-\alpha xy}\tanh(\beta xy)\tilde{f}(x)\tilde{g}(y)\,\mathrm{d}x\,\mathrm{d}y>C(\alpha,\beta)\big(1-o(1)\big)\quad\big(\varepsilon\to0^+\big). \tag{2.11}$$

Proof. We easily get:

$$\tilde{J}\varepsilon=\left[\int_0^\infty x^{-p\beta-1}\tilde{f}^p(x)\,\mathrm{d}x\right]^{\frac{1}{p}}\left[\int_0^\infty y^{-q\beta-1}\tilde{g}^q(y)\,\mathrm{d}y\right]^{\frac{1}{q}}\varepsilon=\left[\int_1^\infty x^{-(1+\varepsilon)}\,\mathrm{d}x\right]^{\frac{1}{p}}\left[\int_0^1 y^{-1+\varepsilon}\,\mathrm{d}y\right]^{\frac{1}{q}}\varepsilon=1.$$

Since $F(u)=u^{\beta+2}\mathrm{e}^{-\frac{\alpha}{\beta}u}\tanh u$ is continuous in $(0,\infty)$ and $\lim_{u\to0}F(u)=0$, $\lim_{u\to\infty}F(u)=0$, there exists $M>0$, satisfying $F(u)\leqslant M$. By Fubini's theorem [3], we have:

$$\begin{aligned}
\tilde{I}\varepsilon&=\varepsilon\int_0^\infty\int_0^\infty \mathrm{e}^{-\alpha xy}\tanh(\beta xy)\tilde{f}(x)\tilde{g}(y)\,\mathrm{d}x\,\mathrm{d}y\\
&=\varepsilon\int_1^\infty x^{\frac{p\beta-\varepsilon}{p}}\,\mathrm{d}x\left[\int_0^1 \mathrm{e}^{-\alpha xy}\tanh(\beta xy)y^{\frac{q\beta+\varepsilon}{q}}\,\mathrm{d}y\right]\\
&=\frac{\varepsilon}{\beta^{\beta+1+\frac{\varepsilon}{q}}}\int_1^\infty x^{-1-\varepsilon}\,\mathrm{d}x\left[\int_0^{\beta x}\mathrm{e}^{-\frac{\alpha}{\beta}u}(\tanh u)u^{\beta+\frac{\varepsilon}{q}}\,\mathrm{d}u\right]\\
&=\frac{\varepsilon}{\beta^{\beta+1+\frac{\varepsilon}{q}}}\int_0^\infty u^{\beta+\frac{\varepsilon}{q}}\mathrm{e}^{-\frac{\alpha}{\beta}u}\tanh u\,\mathrm{d}u-\frac{\varepsilon}{\beta^{\beta+1+\frac{\varepsilon}{q}}}\int_1^\infty x^{-1-\varepsilon}\,\mathrm{d}x\int_{ax}^\infty u^{\beta+\frac{\varepsilon}{q}}\mathrm{e}^{-\frac{\alpha}{\beta}u}\tanh u\,\mathrm{d}u\\
&=\frac{1}{\beta^{\beta+1+\frac{\varepsilon}{q}}}\Gamma\left(\beta+1+\frac{\varepsilon}{q}\right)\sum_{k=0}^\infty(-1)^k\left\{\frac{1}{(2k+\frac{\alpha}{\beta})^{\beta+1}}-\frac{1}{[2(k+1)+\frac{\alpha}{\beta}]^{\beta+1}}\right\}\\
&\quad-\frac{1}{\beta^{\beta+1+\frac{\varepsilon}{q}}}\int_1^\infty x^{-1-\varepsilon}\,\mathrm{d}x\int_{ax}^\infty u^{\beta+\frac{\varepsilon}{q}}\mathrm{e}^{-\frac{\alpha}{\beta}u}\tanh u\,\mathrm{d}u\\
&>\frac{1}{2^\beta\beta^{\beta+1+\frac{\varepsilon}{q}}}\Gamma\left(\beta+1+\frac{\varepsilon}{q}\right)\left[\sum_{k=0}^\infty(-1)^k\frac{1}{(k+\frac{\alpha}{2\beta})^{\beta+1}}-\frac{1}{2}\left(\frac{2\beta}{\alpha}\right)^{\beta+1}\right]-\frac{M\varepsilon}{\beta^{\beta+1+\frac{\varepsilon}{q}}}\int_1^\infty x^{-1}\,\mathrm{d}x\int_{ax}^\infty u^{-2+\frac{\varepsilon}{q}}\,\mathrm{d}u\\
&=\frac{1}{2^\beta\beta^{\beta+1+\frac{\varepsilon}{q}}}\Gamma\left(\beta+1+\frac{\varepsilon}{q}\right)\left[\sum_{k=0}^\infty(-1)^k\frac{1}{(k+\frac{\alpha}{2\beta})^{\beta+1}}-\frac{1}{2}\left(\frac{2\beta}{\alpha}\right)^{\beta+1}\right]-\frac{M\varepsilon}{\beta^{\beta+1}}\frac{1}{(1-\frac{\varepsilon}{q})^2}\\
&=C(\alpha,\beta)\big(1-o(1)\big)\quad\big(\varepsilon\to0^+\big).\qquad\square
\end{aligned}$$

3. Main results and applications

Theorem 3.1. *If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\alpha,\beta>0$, $\varphi(x)=x^{-p\beta-1}$, $\psi(y)=y^{-q\beta-1}$, $f\in L^p_\varphi(0,\infty)$, $g\in L^q_\psi(0,\infty)$, $\|f\|_{p,\varphi},\|g\|_{q,\varphi}>0$, then we have*:

$$\int_0^\infty\int_0^\infty e^{-\alpha xy}\tanh(\beta xy)f(x)g(y)\,dx\,dy < C(\alpha,\beta)\|f\|_{p,\varphi}\|g\|_{q,\varphi}, \tag{3.1}$$

where the constant factor $C(\alpha,\beta)$ ($C(\alpha,\beta)$ the same as (2.5)) is the best possible.

Proof. By Hölder's inequality [4] and Fubini's theorem and Lemma 2.4, we obtain:

$$\begin{aligned}
I &:= \int_0^\infty\int_0^\infty e^{-\alpha xy}\tanh(\beta xy)f(x)g(y)\,dx\,dy\\
&= \int_0^\infty\int_0^\infty e^{-\alpha xy}\tanh(\beta xy)f(x)g(y)\left[\frac{y^{\frac{\beta}{p}}}{x^{\frac{\beta}{q}}}\right]\left[\frac{x^{\frac{\beta}{q}}}{y^{\frac{\beta}{p}}}\right]dx\,dy\\
&\leqslant \left[\int_0^\infty\int_0^\infty e^{-\alpha xy}\tanh(\beta xy)f^p(x)\frac{y^\beta}{x^{\frac{p\beta}{q}}}\,dx\,dy\right]^{\frac{1}{p}}\left[\int_0^\infty\int_0^\infty e^{-\alpha xy}\tanh(\beta xy)g^q(y)\frac{x^\beta}{y^{\frac{q\beta}{p}}}\,dx\,dy\right]^{\frac{1}{q}}\\
&= \left\{\int_0^\infty \omega(\alpha,\beta,x)f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty \omega(\alpha,\beta,y)g^q(y)\,dy\right\}^{\frac{1}{q}}\\
&= C(\alpha,\beta)\|f\|_{p,\varphi}\|g\|_{q,\varphi}.
\end{aligned} \tag{3.2}$$

If inequality (3.2) keeps the form of an equality, then according to [3] there exist two constants A and B such that they are not all zero and:

$$A\frac{y^\beta}{x^{\frac{p\beta}{q}}}f^p(x) = B\frac{x^\beta}{y^{\frac{q\beta}{p}}}g^q(y) \quad \text{a.e. in } (0,\infty)\times(0,\infty).$$

It follows that $Ax^{-p\beta}f^p(x) = By^{-q\beta}g^q(y)$ a.e. in $(0,\infty)\times(0,\infty)$. Assuming that $A \neq 0$, there exists $y > 0$ such that $x^{-p\beta-1}f^p(x) = [By^{-q\beta}g^q(y)]\frac{1}{Ax}$ a.e. in $x \in (0,\infty)$, which contradicts the fact that $0 < \|f\|_{p,\varphi} < \infty$. Then inequality (3.2) keeps the strict form.

If the constant factor $C(\alpha,\beta)$ of (3.1) is not the best possible one, then exists a positive $K < C(\alpha,\beta)$, such that inequality (3.1) is still valid if we replace $C(\alpha,\beta)$ by K, then by (2.10) and (2.11), we have:

$$C(\alpha,\beta)\big(1-o(1)\big) < K.$$

Letting $\varepsilon \to 0^+$, we get $K \geqslant C(\alpha,\beta)$, which contradicts the fact that $K < C(\alpha,\beta)$; so the constant factor $C(\alpha,\beta)$ of (3.1) is the best possible one. The theorem is proved. □

Theorem 3.2. *If $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $\alpha,\beta > 0$, $\varphi(x) = x^{-p\beta-1}$, $f \in L^p_\varphi(0,\infty)$, $\|f\|_{p,\varphi} > 0$, then we have:*

$$\int_0^\infty y^{\frac{q\beta+1}{q-1}}\,dy\left[\int_0^\infty e^{-\alpha xy}\tanh(\beta xy)f(x)\,dx\right]^p < C^p(\alpha,\beta)\|f\|^p_{p,\varphi}, \tag{3.3}$$

where the constant factor $C^p(\alpha,\beta)$ is the best possible one, and inequality (3.3) is equivalent to inequality (3.1).

Proof. Setting a bounded measurable function as:

$$\big[f(x)\big]_n := \min\{n, f(x)\} = \begin{cases} f(x), & \text{for } f(x) < n,\\ n, & \text{for } f(x) \geqslant n.\end{cases}$$

Since $0 < \|f\|_{p,\varphi} < \infty$, there exists $n_0 \in \mathbf{N}$, such that $0 < \int_{\frac{1}{n}}^n \varphi_p(x)[f(x)]_n^p\,dx < \infty$ $(n \geqslant n_0)$, setting:

$$g_n(y) := y^{\frac{p\beta+1}{q-1}}\left[\int_{\frac{1}{n}}^n e^{-\alpha xy}\tanh(\beta xy)\big[f(x)\big]_n\,dx\right]^{\frac{p}{q}} \quad \left(\frac{1}{n} < y < n, n \geqslant n_0\right),$$

when $n \geqslant n_0$, by (3.1) we find:

$$0<\int_{\frac{1}{n}}^{n}\psi(y)g_n^q(y)\,\mathrm{d}y=\int_{\frac{1}{n}}^{n}y^{\frac{q\beta+1}{q-1}}\left[\int_{\frac{1}{n}}^{n}\mathrm{e}^{-\alpha xy}\tanh(\beta xy)\big[f(x)\big]_n\,\mathrm{d}x\right]^p\mathrm{d}y$$
$$=\int_{\frac{1}{n}}^{n}\int_{\frac{1}{n}}^{n}\mathrm{e}^{-\alpha xy}\tanh(\beta xy)\big[f(x)\big]_n g_n(y)\,\mathrm{d}x\,\mathrm{d}y$$
$$<C(\alpha,\beta)\left\{\int_{\frac{1}{n}}^{n}\varphi(x)\big[f(x)\big]_n^p\,\mathrm{d}x\right\}^{\frac{1}{p}}\left\{\int_{\frac{1}{n}}^{n}\psi(y)g_n^q(y)\,\mathrm{d}y\right\}^{\frac{1}{q}},\tag{3.4}$$

$$0<\int_{\frac{1}{n}}^{n}\psi(y)g_n^q(y)\,\mathrm{d}y<C^p(\alpha,\beta)\int_0^\infty\varphi(x)f^p(x)\,\mathrm{d}x=C^p(\alpha,\beta)\|f\|_{p,\varphi}^p<\infty.\tag{3.5}$$

It follows $0<\|f\|_{p,\varphi}<\infty$. For $n\to\infty$, by (3.1), both (3.4) and (3.5) still keep the form of strict inequalities. Hence, we have Eq. (3.3).

On the other hand, by Hölder's inequality, we find:

$$I=\int_0^\infty\int_0^\infty\mathrm{e}^{-\alpha xy}\tanh(\beta xy)f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y=\int_0^\infty\left[y^{\frac{q\beta+1}{p(q-1)}}\int_0^\infty\mathrm{e}^{-\alpha xy}\tanh(\beta xy)f(x)\,\mathrm{d}x\right]\left[y^{\frac{-q\beta-1}{p(q-1)}}g(y)\right]\mathrm{d}y$$
$$\leqslant\left\{\int_0^\infty y^{\frac{q\beta+1}{q-1}}\,\mathrm{d}y\left[\int_0^\infty\mathrm{e}^{-\alpha xy}\tanh(\beta xy)f(x)\,\mathrm{d}x\right]^p\right\}^{\frac{1}{p}}\|g\|_{q,\varphi}<C(\alpha,\beta)\|f\|_{p,\varphi}\|g\|_{q,\varphi}.$$

The inequality is (3.1), which is equivalent to (3.3).

If the constant factor in (3.3) is not the best one, so, by (3.3), the constant factor in (3.1) is not the best one too; this and the conclusion of Theorem 3.1 are contradictory. Thus the constant factor $C^p(\alpha,\beta)$ in (3.3) is the best possible one. □

By taking special parameter values in (3.1) and (3.3), some meaningful inequalities are obtained:

Example 3.1. Let $\alpha=\beta=1$, $p=q=2$, by (2.6), we get $C(1,1)=2\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}-1=2c_0-1$, where c_0 is Catalan's constant. If $\varphi(x)=x^{-3}$, $\psi(y)=y^{-3}$, $f\in L_\varphi^2(0,\infty)$, $g\in L_\psi^2(0,\infty)$, $\|f\|_{2,\varphi},\|g\|_{2,\psi}>0$, then we have (1.6) and its equivalent form as:

$$\int_0^\infty y^3\,\mathrm{d}y\left[\int_0^\infty\mathrm{e}^{-xy}\tanh(xy)f(x)\,\mathrm{d}x\right]^2<(2c_0-1)^2\|f\|_{2,\varphi}^2,\tag{3.6}$$

where the constant factor $(2c_0-1)^2$ is the best possible one.

Example 3.2. Let $\alpha=2$, $\beta=1$, $p=q=2$, by (2.8), we get $C(2,1)=\frac{1}{4}(\frac{\pi^2}{6}-1)$. If $\varphi(x)=x^{-3}$, $\psi(y)=y^{-3}$, $f\in L_\varphi^2(0,\infty)$, $g\in L_\psi^2(0,\infty)$, $\|f\|_{2,\varphi},\|g\|_{2,\psi}>0$, then we have the following equivalent inequality:

$$\int_0^\infty\int_0^\infty\mathrm{e}^{-2xy}\tanh(xy)f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y<\frac{1}{4}\left(\frac{\pi^2}{6}-1\right)\|f\|_{2,\varphi}\|g\|_{2,\psi},\tag{3.7}$$

$$\int_0^\infty y^3\,\mathrm{d}y\left[\int_0^\infty\mathrm{e}^{-2xy}\tanh(xy)f(x)\,\mathrm{d}x\right]^2<\frac{1}{16}\left(\frac{\pi^2}{6}-1\right)^2\|f\|_{2,\varphi}^2,\tag{3.8}$$

where the constant factors $\frac{1}{4}(\frac{\pi^2}{6}-1)$, $\frac{1}{16}(\frac{\pi^2}{6}-1)^2$ in (3.7) and (3.8) are the best possible ones.

Example 3.3. Let $\alpha=\beta=2$, $p=q=2$, by (2.9), we get $C(2,2)=\frac{1}{4}(\frac{5\pi^5}{768}-1)$. If $\varphi(x)=x^{-5}$, $\psi(y)=y^{-5}$, $f\in L_\varphi^2(0,\infty)$, $g\in L_\psi^2(0,\infty)$, $\|f\|_{2,\varphi},\|g\|_{2,\psi}>0$, then we have the following equivalent inequality:

$$\int_0^\infty \int_0^\infty e^{-2xy} \tanh(2xy) f(x) g(y)\, dx\, dy < \frac{1}{4}\left(\frac{5\pi^5}{768} - 1\right) \|f\|_{2,\varphi} \|g\|_{2,\psi}, \tag{3.9}$$

$$\int_0^\infty y^5\, dy \left[\int_0^\infty e^{-2xy} \tanh(2xy) f(x)\, dx\right]^2 < \frac{1}{16}\left(\frac{5\pi^5}{768} - 1\right)^2 \|f\|_{2,\varphi}^2, \tag{3.10}$$

where the constant factors $\frac{1}{4}(\frac{5\pi^5}{768} - 1)$, $\frac{1}{16}(\frac{5\pi^5}{768} - 1)^2$ in (3.9) and (3.10) are the best possible ones.

References

[1] G.H. Hardy, J.E. Littlewood, G. Polya, Inequalities, Cambridge University Press, Cambridge, UK, 1952.
[2] Zhushi Huang, Duiren Guo, An Introduction to Special Function, Beijing Press, Beijing, China, 2000.
[3] Jichang Kuang, Introduction to Real Analysis, Hunan Education Press, Changsha, China, 1996.
[4] Jichang Kuang, Applied Inequalities, 3rd ed., Shandong Science and Technology Press, Jinan, China, 2004.
[5] Jimeng Li, Qiong Liu, A generalization of the Hardy–Hilbert's inequality and its application, Acta Math. Sin. (Chin. Ser.) 52 (2) (2009) 237–244.
[6] Qiong Liu, Shunchao Long, A Hilbert-type integral inequality with the kernel of hyperbolic secant function, J. Zhejiang Univ. Sci. Ed. 40 (3) (2013) 255–259.
[7] D.S. Mintrinovic, J.E. Pecaric, A.M. Kink, Inequalities Involving Functions and Their Integrals and Derivatives, Kluwer Academic Publishers, Boston, 1991.
[8] Bianping Shu, Dongli Chen, Complex-Variable Function and Integral Transform, Higher Education Press, 2003.
[9] Bicheng Yang, The norm of a Hilbert-type linear operator and applications, J. Math. Anal. Appl. 325 (2007) 529–541.
[10] Bicheng Yang, A survey of the study of Hilbert-type inequalities with parameters, Adv. Math. 38 (3) (2009) 257–258.
[11] Bicheng Yang, Hilbert-type integral inequality with non-homogeneous kernel, J. Shanghai Univ. Nat. Sci. 17 (5) (2011) 603–605.