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Partial Differential Equations

On the planning problem for a class of Mean Field Games

Sur le problème de planification pour une classe de jeux à champ moyen

Alessio Porretta

Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica 1, 00133 Roma, Italy

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ABSTRACT

We give a result of existence and uniqueness of weak solutions to the planning problem for a class of Mean Field Games. This is a kind of optimal transportation problem consisting in the exact controllability at time T of Fokker–Planck equations obtained using drifts arising as the optimal feedbacks from a coupled backward Hamilton–Jacobi–Bellman equation.

RÉSUMÉ

Nous donnons un résultat d'existence et d'unicité des solutions faibles du problème de planification pour une classe de jeux à champ moyen. Il s'agit d'un problème de transport optimal qui consiste en la contrôlabilité exacte au temps T de l'équation de Fokker–Planck en utilisant des champs obtenus comme loi feedback optimale d'une équation de Hamilton–Jacobi–Bellman couplée.

Version française abrégée

La théorie des jeux à champ moyen a été développée depuis 2006 par Jean-Michel Lasry et Pierre-Louis Lions [4–8]. Ces auteurs ont introduit une approche de type champ moyen pour des jeux avec un nombre infini (continuum) de joueurs, dont les stratégies dépendent des densités empiriques. Dans la limite où le nombre de joueurs tend vers l'infini, le modèle est décrit par un système couplé d'équations aux dérivées partielles, une équation de Bellman (pour la fonction valeur du joueur générique) et une équation de Fokker–Planck (pour la densité de joueurs). Dans cette Note, nous allons étudier un tel système où *les densités initiale et finale sont prescrites*, voir (1)–(2), où m_0 et m_1 sont des densités de probabilité qui seront supposées positives et régulières. Ce problème a été tout d'abord étudié par P.-L. Lions dans ses conférences [8] comme un modèle de planification pour lequel on souhaite que, quand les agents suivent leur stratégie optimale, leur densité évolue d'une configuration initiale m_0 vers une cible finale m_1 à l'horizon T.

Très peu des résultats sont disponibles pour le problème de planification (1)–(2). P.-L. Lions a prouvé [8] l'existence et l'unicité de solutions régulières si le Hamiltonien est purement quadratique ($H(x,p) = \frac{|p|^2}{2}$, avec éventuellement une perturbation négligeable à l'infini). Dans la stratégie de cette démonstration, le point clé est un changement de fonctions inconnues, qui conduit à un système de deux équations paraboliques semi-linéaires avec des potentiels couplés. D'autres résultats sur des approximations discrètes de (1)–(2) ont été démontrés dans [1].

Dans cette Note, nous présentons un résultat d'existence et d'unicité des solutions faibles de (1)–(2) si le hamiltonien est à croissance quadratique, mais sans supposer un comportement asymptotique précis à l'infini. Comme dans [8], le rôle des

E-mail address: porretta@mat.uniroma2.it.

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conditions aux limites est en quelque sorte négligé en supposant un régime périodique, c'est-à-dire que $F(x,m)$ et $H(x,p)$ sont $\mathbb{Z}^N$-periodiques en x, alors que $\Omega = [0,1]^N$. On suppose, de plus, que – voir (5)–(6) – F est non décroissante par rapport à m et que H est convexe par rapport à p. Ces hypothèses sont cruciales pour la stabilité du système. Enfin, aucune limitation n'est supposée sur la croissance de F par rapport à m, tandis que nous faisons les hypothèses (7)–(9) sur H, qui incluent des croissances de type quadratique générales (par exemple non homogènes par rapport à x). Cependant, par rapport aux résultats de P.-L. Lions, nous sommes seulement en mesure de prouver l'existence de solutions *faibles*. La question de la régularité de ces solutions dans le cadre choisi ici est ouverte. Par ailleurs, nous démontrons que les solutions faibles sont uniques, ce qui donne enfin une caractérisation du problème.

Théorème 1. *Sous les hypothèses (5)–(9), et si $m_0, m_1 \in C^1(\overline{\Omega})$, $m_0, m_1 > 0$, $\int_\Omega m_0 \, dx = \int_\Omega m_1 \, dx = 1$, il existe une solution faible (u,m) du problème (1)–(2). Si, de plus, $H(x,\cdot)$ est strictement convexe, alors la solution faible (u,m) est caractérisée de façon unique ; précisément, si (u_1,m_1) et (u_2,m_2) sont deux solutions faibles de (1)–(2), alors $m_1 = m_2$ et $u_1 = u_2 + c$, $c \in \mathbb{R}$.*

Notre approche pour résoudre le problème ne s'appuie que sur des méthodes d'énergie et de compacité et sur la stabilité du système. En fait, on obtient une solution comme limite de problèmes de jeux à champ moyen standards en pénalisant le coût final, c'est-à-dire en considérant la limite, pour $\varepsilon \to 0$, des problèmes (3). Cette stratégie, déjà exploitée pour le régime discret proposé dans [1], est naturelle du point de vue du contrôle optimal. En effet, le problème de planification (1)–(2) peut être réformulé comme un problème de transport optimal pour l'équation de Fokker–Planck (voir (4)), comparé à la formulation du problème de transfert de masse de Monge–Kantorovich développée dans [2].

Aussi bien l'existence que l'unicité des solutions dans le Théorème 1 nécessitent plusieurs étapes et arguments. Dans cette Note, nous donnons les idées principales pour les deux aspects et les outils qui conduisent au résultat. La preuve détaillée concernant l'existence est contenue dans [9], tandis que l'unicité fait partie d'une étude générale [10] consacrée à des propriétés des solutions faibles de l'équation de Fokker–Planck aussi bien qu'à ses applications à divers problèmes de jeux à champ moyen.

1. Introduction, main result and comments

Since 2006, Jean-Michel Lasry and Pierre-Louis Lions introduced and developed the theory of Mean Field Games, namely a mean field approach to games with infinite number of indistinguishable agents, whose strategies depend on the behavior of the mass of co-players. We refer to [4–8] for an introduction to this theory and a discussion of several concrete models and potential applications in finance and economics.

In the continuum limit, as the number of players tends to infinity, the model is described by a coupled system of PDEs, a Bellman equation (for the value function of the generic player) and a Fokker–Planck equation, for the evolution of the density of players. In its simplest example, this system takes the form:

$$\begin{cases} -u_t - \Delta u + H(x, Du) = F(x,m) & \text{in } (0,T) \times \Omega, \\ m_t - \Delta m - \operatorname{div}\big(m H_p(x, Du)\big) = 0 & \text{in } (0,T) \times \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $H(x,p)$ is the Hamiltonian and $F(x,m)$ is the coupling in the cost criterion.

The Mean Field Games system (1) is usually complemented with a given terminal pay-off $u(T)$ for the value function and an initial condition $m(0)$ for the density of players. In this Note, we will instead study the case where *initial and terminal conditions are prescribed for the density m*, namely:

$$m(0) = m_0, \qquad m(T) = m_1, \tag{2}$$

where m_0, m_1 are probability densities, which will be assumed positive and smooth.

This problem was suggested and firstly studied by P.-L. Lions in his lectures [8] and represents an interesting planning model, in which one wants to drive the density of the agents from an initial configuration m_0 to a target final one $m(T)$ in a way which is still optimal for the agents' strategy. To this extent, it should be considered as an optimal transportation problem; the final payoff $u(T)$, which is not prescribed, should be meant as a degree of control used to achieve the goal to bring the agents toward a final configuration $m(T)$.

Very few results are available for the planning problem (1)–(2). P.-L. Lions proved (see [8]) the existence and uniqueness of *smooth* solutions if the Hamiltonian is purely quadratic ($H(x,p) = \alpha|p|^2$) or, possibly, close at infinity to a purely quadratic one. In his strategy, the key point is given by a change of unknown which reduces (1) to a system of two semilinear parabolic equations with coupled potentials. Furthermore, some results about discrete schemes for (1)–(2) were proved in [1].

In this Note, we present a result of existence and uniqueness of *weak* solutions to (1)–(2) assuming that the Hamiltonian has quadratic growth but no precise asymptotics is required at infinity. Our approach to solve problem (1)–(2) only relies on energy methods and compactness and stability of the system, since we obtain a solution as limit of MFG problems with penalized terminal pay-off:

$$\begin{cases} -(u_\varepsilon)_t - \Delta u_\varepsilon + H(x, Du_\varepsilon) = F(x, m_\varepsilon) & \text{in } Q_T, \\ (m_\varepsilon)_t - \Delta m_\varepsilon - \operatorname{div}\big(m_\varepsilon H_p(x, Du_\varepsilon)\big) = 0 & \text{in } Q_T, \\ m_\varepsilon(0) = m_0, \quad u_\varepsilon(T) = \dfrac{m_\varepsilon(T) - m_1}{\varepsilon} \end{cases} \tag{3}$$

having set $Q_T = (0, T) \times \Omega$. This strategy, already exploited for the discrete scheme proposed in [1], is very natural in the optimal control viewpoint. In fact, as already remarked in [6], (1) can be expressed as the optimality system associated with a control problem on the Fokker–Planck equation, to be seen as the state equation. In particular, the planning problem (1)–(2) may be rephrased as an optimal transport problem for the Fokker–Planck equation:

$$\min_{\alpha \in L^2(m\,\mathrm{d}x\,\mathrm{d}t)} \int_0^T \int_\Omega \big\{L(x, \alpha)m + \Phi(x, m)\big\}\,\mathrm{d}x\,\mathrm{d}t, \qquad \begin{cases} m_t - \Delta m - \operatorname{div}(\alpha m) = 0, \\ m(0) = m_0, \quad m(T) = m_1 \end{cases} \tag{4}$$

with $\Phi(x, m) = \int_0^m F(x, r)\,\mathrm{d}r$ and $H(x, p) = \sup_\alpha [p \cdot \alpha - L(x, \alpha)]$, whereas the approximation (3) corresponds to a (penalized) optimal control problem. The reader should compare (4) to the fluid mechanics formulation of the Monge–Kantorovich mass transfer problem (see [2]).

Let us now be more precise concerning the assumptions and statement of the results. In order to simplify the framework, at this stage we somehow neglect the role of boundary conditions, assuming to be in a periodic setting with period, e.g., $\Omega = [0, 1]^N$. In particular, we assume that:

$$F \in C^0\big(\mathbb{R}^N \times \mathbb{R}\big) \quad \text{is } \mathbb{Z}^N\text{-periodic w.r.t. } x \text{ and nondecreasing with respect to } m, \tag{5}$$

$$H \in C^1\big(\mathbb{R}^N \times \mathbb{R}^N\big) \quad \text{is } \mathbb{Z}^N\text{-periodic w.r.t. } x \text{ and convex with respect to } p. \tag{6}$$

Note that no assumption from above is made on the function F, which may grow arbitrarily fast with respect to m. Concerning the Hamiltonian, we assume that H satisfies, for every $(x, p) \in \Omega \times \mathbb{R}^N$, and for some constants $\alpha, \beta, \gamma_0 > 0$,

$$H_p(x, p) \cdot p - 2H(x, p) \geqslant -\gamma_0, \tag{7}$$

$$\big|H_p(x, p)\big| \leqslant \beta\big(1 + |p|\big), \tag{8}$$

$$H(x, p) \geqslant \alpha|p|^2 - \gamma_0. \tag{9}$$

Our assumptions (7)–(9) aim at extending the previous results to general quadratic growths. Unfortunately, compared with the results by P.-L. Lions, we are only able to prove the existence of *weak* solutions. On the one hand, it is an interesting open problem to understand whether solutions are *smooth* under this more general set of assumptions. On the other hand, we are able to prove that weak solutions are themselves unique, which finally provides us with a characterization of (1)–(2), even if in a weak setting.

Definition 1. A couple (u, m) is a weak solution to (1)–(2) if $m \in C^0([0, T]; L^1(\Omega))$, $\int_\Omega m(t)\,\mathrm{d}x = 1$ for every $t \in [0, T]$, $m(0) = m_0$ and $m(T) = m_1$ in $L^1(\Omega)$, if $u \in L^2(0, T; H^1(\Omega))$ and $m|Du|^2 \in L^1(Q_T)$, $H(x, Du), F(x, m) \in L^1(Q_T)$, and the equations hold in the sense of distributions.

Our result on the planning problem reads as follows.

Theorem 2. *Let $m_0, m_1 \in C^1(\overline{\Omega})$, $m_0, m_1 > 0$, $\int_\Omega m_0\,\mathrm{d}x = \int_\Omega m_1\,\mathrm{d}x = 1$. Assume that (5)–(9) hold true. Then, there exists a weak solution (u, m) of the planning problem (1)–(2).*

If in addition, $H(x, \cdot)$ is strictly convex, then the weak solution (u, m) is uniquely characterized, i.e. if (u_1, m_1) and (u_2, m_2) are weak solutions to (1)–(2), then $m_1 = m_2$ and $u_1 = u_2 + c$ for some $c \in \mathbb{R}$.

Both the existence and the uniqueness aspects in Theorem 2 require several steps and ingredients. The role of this Note is to give the main ideas of both parts. The existence of solutions is obtained passing to the limit as $\varepsilon \to 0$ in (3); to this purpose, we derive new estimates and stability arguments, which we sketch in the next section, the whole detailed proof can be found in [9]. As far as the uniqueness is concerned, this is part of more general results to appear in [10], which involve properties of weak solutions to Fokker–Planck equations as well as natural applications to MFG systems.

2. On the *existence* of solutions

The existence part of Theorem 2 relies on a detailed study of the limit, as ε tends to zero, of the penalized problem (3). This analysis requires two main parts, which may have independent interest.

Part I: estimates. Firstly, as in other exact controllability results, it is crucial to use some kind of observability estimate, which is given here by the following a priori bound.

Lemma 3. *Let* (u, m) *be a solution of* (1). *Then we have*:

$$\int_\Omega |Du(0)|^2 \, dx \leqslant C\left\{ \int_0^T \int_\Omega m|Du|^2 \, dx \, dt + 1 \right\} \tag{10}$$

where $C = C(T, \alpha, \beta, \gamma_0, m_0)$.

The proof of (10) strongly relies on the structure of Hamiltonian system remarked in [3]. Thanks to (10), one can close the estimates and obtain uniform bounds for the solution $(u_\varepsilon, m_\varepsilon)$ of (3), namely:

$$\sup_{t\in[0,T]} \|u_\varepsilon(t)\|_{L^2(\Omega)} + \int_0^T \int_\Omega (m_\varepsilon + 1)|Du_\varepsilon|^2 \, dx \, dt \leqslant C \tag{11}$$

for some constant C independent of ε. Note that, in particular, (11) yields:

$$\|m_\varepsilon(T) - m_1\|_{L^2(\Omega)} \leqslant C\varepsilon. \tag{12}$$

Let us sketch the proof of (11) in order to show the role of Lemma 3. Firstly, we notice the identity:

$$\int_\Omega \big(u_\varepsilon(m_\varepsilon - m_1)\big)_t \, dx + \int_\Omega \{H_p(x, Du_\varepsilon)Du_\varepsilon - H(x, Du_\varepsilon)\}m_\varepsilon \, dx + \int_\Omega F(x, m_\varepsilon)m_\varepsilon \, dx + \int_\Omega m_1 H(x, Du_\varepsilon) \, dx$$
$$= -\int_\Omega Du_\varepsilon Dm_1 \, dx + \int_\Omega F(x, m_\varepsilon)m_1 \, dx.$$

Integrating in $(0, T)$, using (7) and (9) we deduce:

$$\int_\Omega \frac{|m_\varepsilon(T) - m_1|^2}{\varepsilon} \, dx + \alpha \int_0^T \int_\Omega (m_\varepsilon + m_1)|Du_\varepsilon|^2 \, dx \, dt + \int_0^T \int_\Omega F(x, m_\varepsilon)m_\varepsilon \, dx \, dt$$
$$\leqslant -\int_0^T \int_\Omega Du_\varepsilon Dm_1 \, dx \, dt + \int_0^T \int_\Omega F(x, m_\varepsilon)m_1 \, dx \, dt + \int_\Omega u_\varepsilon(0)(m_0 - m_1) \, dx + c.$$

The first two terms on the right-hand side are easily absorbed from the left, since m_1 is $C^1(\overline{\Omega})$ and $m_1 > 0$. On the other hand, the last term is estimated by Poincaré–Wirtinger inequality (we denote $\langle v\rangle = \int_\Omega v \, dx$):

$$\int_\Omega u_\varepsilon(0)(m_0 - m_1) \, dx = \int_\Omega \big(u_\varepsilon(0) - \langle u_\varepsilon(0)\rangle\big)(m_0 - m_1) \, dx \leqslant c \int_\Omega |Du_\varepsilon(0)| \, dx$$

and then estimated thanks to Lemma 3. We conclude that:

$$\int_\Omega \frac{|m_\varepsilon(T) - m_1|^2}{\varepsilon} \, dx + \int_0^T \int_\Omega (m_\varepsilon + 1)|Du_\varepsilon|^2 \, dx \, dt + \int_0^T \int_\Omega F(x, m_\varepsilon)m_\varepsilon \, dx \, dt \leqslant c.$$

Due to (10), this gives a bound for $Du_\varepsilon(0)$ in $L^2(\Omega)$; integrating the equation of u_ε, we also bound $\langle u_\varepsilon(t)\rangle$ uniformly in time, and so we end up with an estimate of u_ε in $L^2(0, T; H^1(\Omega))$ as well as of $u_\varepsilon(0)$ in $H^1(\Omega)$. Thanks to this latter bound, we can now improve the above estimate; looking at $(u_\varepsilon|u_\varepsilon|(m_\varepsilon - m_1))_t$, in a similar way as before we obtain from the coupled equations that:

$$\int_0^T \int_\Omega |u_\varepsilon|(m_\varepsilon + 1)|Du_\varepsilon|^2 \, dx \, dt + \int_\Omega F(x, m_\varepsilon)m_\varepsilon|u_\varepsilon| \, dx \, dt \leqslant c.$$

Since $(\frac{u_\varepsilon^2}{2})_t = \{-\Delta u_\varepsilon + H(x, Du_\varepsilon) - F(x, m_\varepsilon)\}u_\varepsilon$, this yields a uniform bound for $\|u_\varepsilon(t)\|_{L^2(\Omega)}$ and concludes (11). Let us stress that further estimates can be obtained, playing somehow a role in the compactness arguments, but we refer to the whole proof in [9] for this more technical part.

Part II: compactness. The bounds derived so far allow us to deduce the existence, up to subsequences, of some limit (u, m) such that u_ε, m_ε and Du_ε almost everywhere converge to u, m and Du, respectively. Next, we can use the structural stability of MFG systems; indeed, as showed by Lasry and Lions [6,7], for every couple of smooth solutions (u_1, m_1) and (u_2, m_2) of (1), we have:

$$\begin{aligned}&\int_\Omega \big[(u_1-u_2)(m_1-m_2)\big]_t \,\mathrm{d}x + \int_\Omega m_1\big[H(x,Du_2)-H(x,Du_1)-H_p(x,Du_1)(Du_2-Du_1)\big]\,\mathrm{d}x\\ &\quad+\int_\Omega m_2\big[H(x,Du_1)-H(x,Du_2)-H_p(x,Du_2)(Du_1-Du_2)\big]\,\mathrm{d}x\\ &\quad+\int_\Omega \big[F(x,m_1)-F(x,m_2)\big][m_1-m_2]\,\mathrm{d}x = 0.\end{aligned} \tag{13}$$

Applying this equality to $(u_\varepsilon, m_\varepsilon)$ and (u_η, m_η), integrating and using the initial–terminal conditions, and the L^2 bounds (11)–(12), we deduce:

$$\begin{aligned}&\int_0^T\int_\Omega m_\varepsilon\big\{H(x,Du_\eta)-H(x,Du_\varepsilon)-H_p(x,Du_\varepsilon)(Du_\eta-Du_\varepsilon)\big\}\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_0^T\int_\Omega m_\eta\big\{H(x,Du_\varepsilon)-H(x,Du_\eta)-H_p(x,Du_\eta)(Du_\varepsilon-Du_\eta)\big\}\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_0^T\int_\Omega \big(F(x,m_\varepsilon)-F(x,m_\eta)\big)(m_\varepsilon-m_\eta)\,\mathrm{d}x\,\mathrm{d}t \leqslant c(\varepsilon+\eta).\end{aligned}$$

The monotonicity of F and convexity of H, together with some real analysis argument, allow us to deduce from this inequality the convergence of the energy, namely that $m_\varepsilon|Du_\varepsilon|^2 \to m|Du|^2$ in $L^1(Q_T)$. This is enough to pass to the limit in the Fokker–Planck equation, whereas (12) yields the desired final condition. Finally, in order to conclude, one needs to pass to the limit in the HJB equation; this is by far not obvious due to the lack of bounds from below on u_ε. In fact, in the typical stability arguments, the nonlinearity $H(x, Du_\varepsilon)$ can be handled only in case of positive (or bounded below) solutions. We introduce then a new trick adapted to the MFG system, which roughly speaking amounts to obtain the limit of the first equation only in the subset $\{m > 0\}$, showing however that this set has full measure. This relies on the following new estimate, which holds for any weak solution m:

$$\sup_{[0,T]}\big\|\log m(t)\big\|_{L^1(\Omega)} + \|\log m\|^2_{L^2(0,T;H^1(\Omega))} \leqslant c\big\{1+\|\log m_0\|_{L^1(\Omega)} + \big\|H_p(x,Du)\big\|^2_{L^2(Q_T)}\big\}. \tag{14}$$

The estimate (14), somehow reminiscent of the strong maximum principle, ensures that $m > 0$ a.e.; a similar bound, uniform in ε, is obtained for m_ε and allows us to apply successfully our strategy obtaining the equation for u (see [9] for details).

3. On the uniqueness of weak solutions

In itself, the uniqueness principle comes from the original argument found by Lasry and Lions for smooth solutions, which relies on (13); the non-trivial fact is that this proof extends to the general class of weak solutions. To this purpose, we use a renormalization principle; first of all, our approach stands on the following uniqueness result on the Fokker–Planck equations, which has an interest in its own. We denote by $\theta_k(\xi) = \min(1, \frac{(2k-|\xi|)^+}{k})$ a compactly supported approximation of the unit function, then $T_k(s) := \int_0^s \theta_k(\xi)\,\mathrm{d}\xi$ is a C^1 truncation. We still assume that all functions are $\mathbb{Z}^N$-periodic.

Theorem 4. *Let $V \in L^2(Q_T)^N$ and $m_0 \in L^1(\Omega)$. Then the problem:*

$$\begin{cases} m_t - \Delta m - \operatorname{div}(mV) = 0, & \text{in } Q_T,\\ m(0) = m_0 & \text{in } \Omega\end{cases} \tag{15}$$

admits at most one weak solution m such that $m|V|^2 \in L^1(Q_T)$. Moreover, in this case, any weak solution is a renormalized solution and satisfies:

$$\big(T_k(m)\big)_t - \Delta T_k(m) - \operatorname{div}\big(T_k'(m)mV\big) = \omega_k, \quad \text{in } (0,T)\times\Omega, \tag{16}$$

where $\omega_k \in L^1(Q_T)$, and $\omega_k \to^{k\to\infty} 0$ in $L^1(Q_T)$.

Let us stress the relevance of the assumption $m|V|^2 \in L^1(Q_T)$; without this condition, uniqueness could fail for merely L^2 vector fields V. Such condition is also significant in terms of the underlying stochastic dynamics, meaning that the field V is square integrable along the trajectory. Theorem 4 has a fundamental consequence for Mean Field Games systems, since any weak solution (u, m) satisfies $mH_p(x, Du)^2 \in L^1(Q_T)$. Therefore, m is unique and, yet more important, satisfies a renormalized equation. One more ingredient is essential for the uniqueness of weak solutions, which is the following *crossed regularity* lemma.

Lemma 5. *Assume (5)–(9). Given any two weak solutions (u_1, m_1) and (u_2, m_2) of (1)–(2), we have*:

$$F(m_i)m_j \in L^1(Q_T), \qquad m_i|Du_j|^2 \in L^1(Q_T), \quad \forall i, j = 1, 2. \tag{17}$$

Having property (17) in hand, one can, roughly speaking, extend (13) to weak solutions, passing to the limit from the renormalized formulations; once (13) (integrated in $(0, T)$) is obtained, this rapidly leads to the conclusion. For the proofs, and a more precise introduction to renormalized solutions, we refer to [10], where we also discuss the link with previous results on Fokker–Planck equations and give similar applications to Mean Field Games in different settings.

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