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Mathematical Analysis/Differential Geometry

# Spectral properties of Schrödinger operators on compact manifolds: Rigidity, flows, interpolation and spectral estimates



*Propriétés spectrales d'opérateurs de Schrödinger sur des variétés compactes : Rigidité, flots, interpolation et estimations spectrales*

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## ARTICLE INFO

### Article history:

Received 12 March 2013

Accepted 1 July 2013

Available online 31 July 2013

Presented by the Editorial Board

## ABSTRACT

This note is devoted to optimal spectral estimates for Schrödinger operators on compact connected Riemannian manifolds without boundary. These estimates are based on the use of appropriate interpolation inequalities and on some recent rigidity results for nonlinear elliptic equations on those manifolds.

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## R É S U M É

Cette note est consacrée à des estimations spectrales optimales pour des opérateurs de Schrödinger sur des variétés riemanniennes compactes et simplement connexes, sans bord. Ces estimations sont basées sur certaines inégalités d'interpolation et sur un résultat récent de rigidité pour des équations elliptiques non linéaires sur ces variétés.

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## 1. Spectral properties of Schrödinger operators on the sphere

We start by briefly reviewing some results that have been obtained in [4]. Let us define  $2^* := \frac{2d}{d-2}$  if  $d \geq 3$ , and  $2^* := \infty$  if  $d = 1$  or  $2$ . We denote by  $\Delta_g$  the Laplace–Beltrami operator on the unit sphere  $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ . It is well known (see [2]) that the equation:

$$-\Delta_g u + \frac{\lambda}{q-2} u = u^{q-1}$$

has only constant solutions as long as  $q \in (2, 2^*)$  and  $\lambda \leq d$ . See [3] for a review and various related results. Assume that the measure on  $\mathbb{S}^d$  is the one induced by Lebesgue's measure on  $\mathbb{R}^{d+1}$ . This convention differs from the one of [4]. The inequality:

$$\|\nabla u\|_{L^2(\mathbb{S}^d)}^2 + \alpha \|u\|_{L^q(\mathbb{S}^d)}^2 \geq \mu(\alpha) \|u\|_{L^q(\mathbb{S}^d)}^2, \quad \forall u \in H^1(\mathbb{S}^d),$$

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for any  $q \in (2, 2^*)$  can be established by standard variational methods. According to [4], the optimal function  $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is concave, increasing, and such that  $\mu(\alpha) = \kappa\alpha$  for any  $\alpha \leq \frac{d}{q-2}$ ,  $\mu(\alpha) < \kappa\alpha$  for  $\alpha > \frac{d}{q-2}$  where  $\kappa := |\mathbb{S}^d|^{1-2/q}$  is a normalization factor and:

$$\mu(\alpha) \sim K_{q,d} \alpha^{1-\vartheta} \quad \text{as } \alpha \rightarrow +\infty, \quad \text{where } \vartheta := d \frac{q-2}{2q},$$

$$K_{q,d} := \inf_{v \in H^1(\mathbb{R}^d) \setminus \{0\}} \frac{\|\nabla v\|_{L^2(\mathbb{R}^d)}^2 + \|v\|_{L^2(\mathbb{R}^d)}^2}{\|v\|_{L^q(\mathbb{R}^d)}^2}. \tag{1}$$

Let us define  $p = \frac{q}{q-2}$  so that  $p \in (1, +\infty)$  if  $d = 1$  and  $p \in (\frac{d}{2}, +\infty)$  if  $d \geq 2$ . If we denote by  $\mu \mapsto \alpha(\mu)$  the inverse function of  $\alpha \mapsto \mu(\alpha)$  and by  $\lambda_1(-\Delta_g - V)$  the lowest (nonpositive) eigenvalue of  $-\Delta_g - V$ , then we have the estimate

$$|\lambda_1(-\Delta_g - V)| \leq \alpha(\|V\|_{L^p(\mathbb{S}^d)}) \quad \forall V \in L^p(\mathbb{S}^d),$$

for any nonnegative  $V \in L^p(\mathbb{S}^d)$ . Moreover we have  $\alpha(\mu)^{p-d/2} = L_{p-\frac{d}{2},d}^1 \mu^p (1 + o(1))$  as  $\mu \rightarrow +\infty$  where  $L_{\gamma,d}^1 := (K_{q,d})^{-(\gamma+d/2)}$ . Equality is achieved for any  $\mu > 0$  by some nonnegative  $V$ , which is constant if and only if  $\mu \leq \frac{d}{2}(p-1)$ .

The case  $q \in (1, 2)$  can also be covered and we refer to [4] for further details. This case leads to estimates from below for the first eigenvalue of the operator  $-\Delta_g + W$ , where  $W$  is a positive potential.

### 2. A rigidity result on compact manifolds and a subcritical interpolation inequality

From here on we shall assume that  $(\mathfrak{M}, g)$  is a smooth compact connected Riemannian manifold of dimension  $d \geq 1$ , without boundary, and let  $\Delta_g$  be the Laplace–Beltrami operator on  $\mathfrak{M}$ . We shall denote by  $dv_g$  the volume element and by  $\mathfrak{R}$  the Ricci tensor. Let  $\lambda_1$  be the lowest positive eigenvalue of  $-\Delta_g$ .

For such manifolds, a new rigidity result has been recently established in [5], thus extending a series of results of [6,2,9,1,8]. In order to state this result, let us define the quantities:

$$\theta = \frac{(d-1)^2(p-1)}{d(d+2)+p-1} \quad \text{and} \quad Q_g u := H_g u - \frac{g}{d} \Delta_g u - \frac{(d-1)(p-1)}{\theta(d+3-p)} \left[ \frac{\nabla u \otimes \nabla u}{u} - \frac{g}{d} \frac{|\nabla u|^2}{u} \right]$$

where  $H_g u$  denotes Hessian of  $u$ , and

$$\Lambda_\star := \inf_{u \in H^2(\mathfrak{M}) \setminus \{0\}} \frac{(1-\theta) \int_{\mathfrak{M}} (\Delta_g u)^2 dv_g + \frac{\theta d}{d-1} \int_{\mathfrak{M}} [\|Q_g u\|^2 + \mathfrak{R}(\nabla u, \nabla u)] dv_g}{\int_{\mathfrak{M}} |\nabla u|^2 dv_g}. \tag{2}$$

It is not difficult to see that  $\Lambda_\star \leq \lambda_1$ .

**Theorem 1.** (Cf. [5, Theorem 3].) Assume that  $\Lambda_\star$  is strictly positive. Then for any  $q \in (1, 2) \cup (2, 2^*)$  and any  $\lambda \in (0, \Lambda_\star)$ , the equation:

$$-\Delta_g v + \frac{\lambda}{q-2} (v - v^{q-1}) = 0$$

has 1 as its unique positive solution in  $C^2(\mathfrak{M})$ .

Note that in the particular case  $\mathfrak{M} = \mathbb{S}^d$ ,  $\Lambda_\star = \lambda_1(-\Delta_g) = d$ . The proof relies on the nonlinear flow

$$u_t = u^{2-2\beta} \left( \Delta_g u + (1 + \beta(q-2)) \frac{|\nabla u|^2}{u} \right), \quad \beta = \frac{(d+2)(d+3-p)\theta}{(d-1)^2(p-1)^2 - (d+2)^2(p-2)\theta}, \tag{3}$$

that can also be used to prove the following  $A$ - $B$ -type interpolation inequality (see [2,7]). Let us define:

$$\kappa := \text{vol}_g(\mathfrak{M})^{1-2/q}.$$

**Theorem 2.** (Cf. [5, Theorem 4].) For any  $q \in (1, 2) \cup (2, 2^*)$  if  $d \geq 3$ ,  $q \in (1, 2) \cup (2, \infty)$  if  $d = 1$  or 2, the inequality:

$$\|\nabla v\|_{L^2(\mathfrak{M})}^2 \geq \frac{\Lambda}{q-2} [\kappa \|v\|_{L^q(\mathfrak{M})}^2 - \|v\|_{L^2(\mathfrak{M})}^2] \quad \forall v \in H^1(\mathfrak{M})$$

holds for some optimal  $\Lambda \in [\Lambda_\star, \lambda_1]$  if  $\Lambda_\star > 0$ . Moreover, if  $\Lambda_\star < \lambda_1$ , then we have  $\Lambda_\star < \Lambda \leq \lambda_1$ .

The above results hold true because the flow (3) contracts:

$$\mathcal{F}[u] := \int_{\mathfrak{M}} |\nabla(u^\beta)|^2 dv_g + \frac{\Lambda_\star}{q-2} \left[ \int_{\mathfrak{M}} u^{2\beta} dv_g - \kappa \left( \int_{\mathfrak{M}} u^{\beta q} dv_g \right)^{2/q} \right].$$

The above choices for  $\theta$  and  $\beta$  are optimal for this contraction property: see [5].

As a consequence and exactly as in the case of the sphere, we get the first result of this note.

**Proposition 3.** Assume that  $q \in (2, 2^*)$  if  $d \geq 3$ , or  $q \in (2, \infty)$  if  $d = 1$  or  $2$ . There exists a concave increasing function  $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that  $\mu(\alpha) = \kappa\alpha$  for any  $\alpha \leq \frac{\Lambda_\star}{q-2}$ ,  $\mu(\alpha) < \kappa\alpha$  for  $\alpha > \frac{\Lambda_\star}{q-2}$  and

$$\|\nabla u\|_{L^2(\mathfrak{M})}^2 + \alpha \|u\|_{L^2(\mathfrak{M})}^2 \geq \mu(\alpha) \|u\|_{L^q(\mathfrak{M})}^2 \quad \forall u \in H^1(\mathfrak{M}).$$

The asymptotic behavior of  $\mu$  is given by  $\mu(\alpha) \sim K_{q,d} \alpha^{1-\vartheta}$  as  $\alpha \rightarrow +\infty$ , with  $\vartheta = d \frac{q-2}{2q}$  and  $K_{q,d}$  defined by (1).

**Proof.** There is an optimal function for the interpolation inequality, as can be shown by standard variational techniques. Applying Theorem 1 to the solutions of the Euler–Lagrange equations completes the proof for fixed values of  $\alpha$ . As an infimum on  $u$  of affine functions with respect to  $\alpha$ , the function  $\alpha \rightarrow \mu(\alpha)$  is concave. It remains to establish the properties of  $\alpha$  for large values of  $\mu$ .

Using a well-chosen test function obtained by scaling an optimal function for (1) on the tangent plane to an arbitrary point of  $\mathfrak{M}$ , one can prove that  $\limsup_{\alpha \rightarrow +\infty} \alpha^{\vartheta-1} \mu(\alpha) \leq K_{q,d}$ . Arguing by contradiction as in [4, Proposition 10], we can find a sequence  $(\alpha_n)_{n \in \mathbb{N}}$  such that  $\lim_{n \rightarrow +\infty} \alpha_n = +\infty$  and  $\lim_{n \rightarrow +\infty} \alpha_n^{\vartheta-1} \mu(\alpha_n) < K_{q,d}$ , and a sequence of optimal functions  $(u_n)_{n \in \mathbb{N}}$  such that  $\|u_n\|_{L^q(\mathbb{S}^d)} = 1$ , which concentrates because  $\limsup_{n \rightarrow +\infty} \alpha_n^{\vartheta} \|u_n\|_{L^2(\mathbb{S}^d)}^2 < K_{q,d}$ . Some classical surgery and a convexity inequality provide a contradiction by constructing a minimizing sequence for (1).  $\square$

### 3. Ground state estimates for Schrödinger operators on Riemannian manifolds

In this section, we keep using the notations of Section 2 and generalize to  $(\mathfrak{M}, g)$  the spectral results established for the sphere in [4]. By inverting the function  $\alpha \mapsto \mu(\alpha)$ , we see that  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is increasing, convex and satisfies:  $\alpha(\mu) = \frac{\mu}{\kappa}$  for any  $\mu \in (0, \frac{\kappa\Lambda_\star}{q-2})$ ,  $\alpha(\mu) > \frac{\mu}{\kappa}$  for  $\mu \in (\frac{\kappa\Lambda_\star}{q-2}, +\infty)$ . With  $L_{\gamma,d}^1 := (K_{q,d})^{-p}$ ,  $\gamma = p - \frac{d}{2}$ , we obtain for a general manifold  $\mathfrak{M}$  the same behavior of  $\mu \mapsto \alpha(\mu)$  when  $\mu \rightarrow +\infty$  as in the case of a sphere.

Let  $\mu := \|V\|_{L^p(\mathbb{S}^d)}$ . Since  $p$  and  $\frac{q}{2}$  are Hölder conjugate exponents, it follows from Hölder’s inequality that:

$$\int_{\mathfrak{M}} |\nabla u|^2 dv_g - \int_{\mathfrak{M}} V|u|^2 dv_g + \alpha(\mu) \int_{\mathfrak{M}} |u|^2 dv_g \geq \|\nabla u\|_{L^2(\mathbb{S}^d)}^2 - \mu \|u\|_{L^q(\mathbb{S}^d)}^2 + \alpha(\mu) \|u\|_{L^2(\mathbb{S}^d)}^2$$

with equality if  $V^{p-1}$  and  $|u|^2$  are proportional. The right-hand side is nonnegative according to Proposition 3. By taking the infimum of the left-hand side, we can deduce an estimate of the lowest, nonpositive eigenvalue  $\lambda_1(-\Delta_g - V)$  of  $-\Delta_g - V$ , which provides us with our first main result.

**Theorem 4.** Let  $d \geq 1$ ,  $p \in (1, +\infty)$  if  $d = 1$  and  $p \in (\frac{d}{2}, +\infty)$  if  $d \geq 2$  and assume that  $\Lambda_\star > 0$ . With the above notations and definitions, for any nonnegative  $V \in L^p(\mathfrak{M})$ , we have:

$$|\lambda_1(-\Delta_g - V)| \leq \alpha(\|V\|_{L^p(\mathfrak{M})}). \tag{4}$$

Moreover, we have  $\alpha(\mu)^{p-\frac{d}{2}} = L_{p-\frac{d}{2},d}^1 \mu^p (1 + o(1))$  as  $\mu \rightarrow +\infty$ .

The estimate (4) is optimal in the sense that for any  $\mu \in (0, +\infty)$ , there exists a nonnegative function  $V$  such that  $\mu = \|V\|_{L^p(\mathfrak{M})}$  and  $|\lambda_1(-\Delta_g - V)| = \alpha(\mu)$ . Moreover, if  $\mu < \frac{\kappa\Lambda_\star}{q-2}$ ,  $\alpha(\mu) = \frac{\mu}{\kappa}$  and equality in (4) is achieved by constant potentials.

In the case of operators  $-\Delta_g + W$  on  $\mathfrak{M}$ , where  $W$  is a nonnegative potential, following again the same arguments as in [4] in the case of the sphere, and the rigidity result of Theorem 1, we obtain our second main result.

**Theorem 5.** Let  $d \geq 1$ ,  $p \in (0, +\infty)$ . There exists an increasing concave function  $v : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , satisfying  $v(\beta) = \beta/\kappa$ , for any  $\beta \in (0, \frac{p+1}{2}\kappa\Lambda)$  if  $p > 1$ , such that for any positive potential  $W$  we have:

$$\lambda_1(-\Delta + W) \geq v(\beta) \quad \text{with} \quad \beta = \left( \int_{\mathfrak{M}} W^{-p} dv_g \right)^{1/p}.$$

Moreover, for large values of  $\beta$ , we have  $v(\beta)^{-(p+\frac{d}{2})} = L_{-(p+\frac{d}{2}),d}^1 \beta^{-p} (1 + o(1))$  as  $\beta \rightarrow +\infty$ .

With  $p = \frac{q}{2-q}$ , the spectral estimate of [Theorem 5](#) is derived from the interpolation inequality:

$$\|\nabla u\|_{L^2(\mathfrak{M})}^2 + \beta \left( \int_{\mathfrak{M}} |u|^q dv_g \right)^{2/q} \geq v(\beta) \|u\|_{L^2(\mathfrak{M})}^2 \quad \forall u \in H^1(\mathfrak{M}).$$

The concentration phenomena leading to the asymptotics for large norms of  $W$  can be studied as in [Proposition 3](#): see [\[4\]](#) for the proof in the case of a sphere. We omit the details of the proof of [Theorem 5](#).

### Acknowledgements

J.D. and M.J.E. have been partially supported by ANR grants *CBDif* and *NoNAP*. They thank the Mittag-Leffler Institute, where part of this research was carried out, for hospitality. M.L. was supported in part by NSF grant DMS-0901304.

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