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Differential Topology

A proof of Morse's theorem about the cancellation of critical points



Une preuve du théorème de Morse sur l'élimination de points critiques

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ABSTRACT

In this Note, we give a proof of the famous theorem of M. Morse dealing with the cancellation of a pair of non-degenerate critical points of a smooth function. Our proof consists of a reduction to the one-dimensional case where the question becomes easy to answer.

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RÉSUMÉ

Dans cette Note, nous présentons une preuve du célèbre théorème de M. Morse concernant l'élimination d'une paire de points critiques non dégénérés pour une fonction C^∞ sur une variété différentiable. Notre preuve consiste à réduire la question au cas facile d'une fonction d'une variable.

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Version française abrégée

Considérons une variété différentiable M et une fonction $f : M \rightarrow \mathbb{R}$ de classe C^∞ à points critiques non dégénérés (fonction de Morse). Marston Morse donna un critère pour l'élimination d'une paire de points critiques d'indices consécutifs. Il le fit d'abord pour une paire d'indices $(0, 1)$ (voir [6]). Puis il généralisa le critère à une paire de points critiques d'indices $(k, k + 1)$ (voir [7] et aussi [2]). S. Smale donna une forme plus algébrique à ce critère, et en fit un outil essentiel dans sa preuve de la conjecture de Poincaré généralisée [10]. Une preuve détaillée du critère de Morse est donnée dans le livre de J. Milnor, *Lectures on the h-Cobordism Theorem* [5].

La preuve donnée dans ce livre passe par l'élimination des deux zéros correspondants du champ de gradient et elle cache la déformation de la fonction à travers la strate de codimension 1 des fonctions ayant une singularité cubique. En particulier, le support de la déformation de la fonction reste invisible. Les notes de J. Cerf et A. Gramain [1] donnent une approche fonctionnelle qui « majore » ce support ; la déformation n'y est pas encore totalement explicite. Notre méthode, qui semble la plus naturelle, fournit un support de déformation qui ne nécessite pas la *saturation* apparaissant dans le texte de Cerf et Gramain.

On équipe f d'un *pseudo-gradient* (descendant) X , champ de vecteurs vérifiant $X \cdot f < 0$ en dehors du lieu critique $\text{crit } f$ et une condition de non-dégénérescence aux points critiques. Cette condition implique que chaque zéro p de X

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est hyperbolique. Il existe donc des variétés stables et instables, notées respectivement $W^s(p)$ et $W^u(p)$. Voici le critère principal d'élimination ; d'autres énoncés s'en déduisent aisément.

Théorème. Soit p et q deux points critiques vérifiant les deux conditions ci-dessous :

- i) $W^u(p)$ et $W^s(q)$ se rencontrent transversalement et leur intersection contient une seule orbite ℓ de X ;
- ii) pour $\varepsilon > 0$ assez petit, les orbites de X contenues dans $W^u(p)$ et distinctes de ℓ coupent le niveau $f^{-1}(f(q) - \varepsilon)$.

Dans ces conditions, la paire (p, q) est éliminable. Plus précisément, si U désigne un voisinage ouvert de l'adhérence de $W^u(p) \cap \{f \geq f(q) - \varepsilon\}$, il existe un chemin de fonctions $(f_t)_{t \in [0,1]}$ tel que :

- 1) $f_0 = f$;
- 2) pour tout $t \in [0, 1]$, f_t coïncide avec f sur $M \setminus U$;
- 3) la restriction $f_t|_U$ est une fonction de Morse à deux points critiques si $0 \leq t < 1/2$; elle a un point critique cubique pour $t = 1/2$ et elle n'a pas de point critique si $1/2 < t \leq 1$.

Idée de la preuve. On trouve un arc paramétré $\alpha : [0, 1] \rightarrow M$, d'image A , qui vérifie les conditions suivantes où on pose $h := f \circ \alpha$:

- A est contenu dans $W^u(p)$, sauf dans un voisinage de q où A est contenu dans $W^s(q)$;
- A contient la trace de ℓ entre les niveaux $\{f = f(p) - \varepsilon\}$ et $\{f = f(q) + \varepsilon\}$;
- h a un maximum en u_0 et un minimum local en u_1 , où $\alpha(u_0) = p$ et $\alpha(u_1) = q$;
- les valeurs aux extrémités sont $h(0) = f(q) - \varepsilon$ et $h(1) = f(q) + \varepsilon$.

On observe qu'il existe une fonction strictement croissante h_1 vérifiant la condition $(*)$: $h_1 \leq h$. C'est la solution à une variable.

Puis on trouve un voisinage tubulaire N de l'image de α et des coordonnées (u, y, z) tels que :

- $f(u, y, z) = h(u) - |y|^2 + |z|^2$;
- $\alpha([0, 1]) = \{y = 0, z = 0\}$;
- $(W^u(p) \cap N) \subset \{z = 0\}$, sauf éventuellement dans un voisinage de Morse de q .

En effet, près de p il existe des coordonnées de Morse (u, y, z) où u est une coordonnée locale de A et (u, y) sont des coordonnées locales de $W^u(p)$; de même, au voisinage de q , on a des coordonnées de Morse où u paramètre A et (u, z) paramètre $W^s(q)$. Et le long des branches monotones de α , il n'y a pas d'obstruction à prolonger ces coordonnées avec les conditions requises.

Grâce alors à un pseudo-gradient convenable, on prolonge ces coordonnées à tout un voisinage U de $W^u(p)$ tronqué au-dessus du niveau $f = f(q) - \varepsilon$, en satisfaisant toujours les mêmes conditions.

On tranche alors U par $u = \text{const}$. Sur chaque tranche $u = c$, la restriction de f est une fonction de Morse avec un seul point critique en $(y, z) = (0, 0)$, de valeur critique $h(c)$. Un lemme facile de descente d'une valeur critique, appliqué dans chaque tranche et valable dans une famille à un paramètre de fonctions (ici le paramètre est $u \in [0, 1]$), permet de déformer $f|_U$ par une déformation à support dans U , jusqu'à $f_1(u, y, z) = h_1(u) - |y|^2 + |z|^2$, qui n'a pas de point critique. Au cours de la déformation, la restriction à chaque tranche $u = c$ est restée une fonction de Morse. \square

1. Introduction

Let us consider an n -dimensional closed manifold M equipped with a Morse function $f : M \rightarrow \mathbb{R}$. Marston Morse devoted two papers to the question of canceling a pair of critical points. In [6] he considered the possibility of canceling a local minimum (index 0) with a critical point of index 1 and similarly, the cancellation of a pair of indices $(n-1, n)$; in that paper, a *polar* function means a function with no supernumerary local extrema. In [7] (see also [2]), Morse extended its cancellation criterion to the case of a pair of indices $(k, k+1)$. It is worth noticing that these two papers show the first time Morse is considering gradients globally in the manifold M , in the spirit of the 1949 Note by René Thom [11].

S. Smale generalized Morse's criterion and got a more algebraic criterion for cancellation. This generalization became an important tool in Smale's proof of the *generalized Poincaré conjecture* [10]. A detailed proof is offered by J. Milnor in [5], where the cancellation theorem of Morse occupies one chapter (i.e. Section 5) and is said to be "quite formidable".

Actually, Milnor's proof deals with the cancellation of two zeroes of a gradient (or *pseudo-gradient*) allowing the deformation to run among vector fields which *a priori* are not gradients. The effective deformation of the function remains hidden and the crossing of the codimension – one stratum of functions with one cubic singularity is invisible. Our goal is to make this deformation visible; in particular, the support of the deformation will be specified, as it is in the notes by J. Cerf and A. Gramain [1] (actually, we specify a smaller support than in [1], where some *saturation* is not useful). Moreover, we

intend to show that the question of cancellation reduces to the one-dimensional case, where it is easy to solve. The same technique was already used in [3].

The statement we give here is slightly different from the one in [5], which will be derived as a corollary. Moreover, we work with *descending pseudo-gradient*, called *pseudo-gradient* for short. According to K. Meyer [4], it is a vector field X satisfying the two following conditions:

- Lyapunov inequality: $X \cdot f < 0$ apart from the critical points of f ;
- Non-degeneracy condition: $X \cdot f$ has a non-degenerate maximum at each critical point of f .

Such a pseudo-gradient X may be built by partition of unity. It follows from the definition that if p is a critical point of f , it is a hyperbolic zero of X and hence, there are stable and unstable manifolds, respectively denoted by $W^s(p)$ and $W^u(p)$, which are formed by the points $x \in M$ such that $X^t(x)$ tends to p as t tends to $+\infty$ or $-\infty$; here X^t denotes the flow of X . The dimension of the unstable manifold $W^u(p)$ and the codimension of the stable manifold $W^s(p)$ both equal the index of the critical point p . It is easily checked that the restriction of f to $W^u(p)$ (resp. $W^s(p)$) has a non-degenerate maximum (resp. minimum).

Theorem. *Let us consider the Morse function $f : M^n \rightarrow \mathbb{R}$ equipped with a pseudo-gradient X . Let p and q be two critical points of f satisfying the following conditions:*

- i) $W^u(p)$ and $W^s(q)$ intersect transversely and the intersection is made of one orbit ℓ of X only.
- ii) For some $\varepsilon > 0$, each orbit of X in $W^u(p)$ distinct from ℓ crosses the level set $f^{-1}(f(q) - \varepsilon)$.

Then the pair (p, q) is cancelable. More precisely, if U denotes an open neighborhood of the closure of $W^u(p) \cap \{f \geq f(q) - \varepsilon\}$, there is a path of smooth functions $(f_t)_{t \in [0,1]}$, such that:

- 1) $f_0 = f$;
- 2) for every $t \in [0, 1]$, f_t coincides with f on $M \setminus U$;
- 3) the function $f_t|_U$ is Morse with two critical points when $0 \leq t < 1/2$; it has a cubic singularity when $t = 1/2$ and it has no critical point when $1/2 < t \leq 1$.

By computing the dimensions, one has the following formula for the Morse indices: $\text{ind}(p) = \text{ind}(q) + 1$. There are other statements by varying assumption ii) which can be derived from the above theorem.

2. Proof of the theorem

Say $\text{index}(p) = k + 1 = \text{index}(q) + 1$. We are given an open neighborhood U of the closure of $W^u(p) \cap \{f(q) \leq f \leq f(p)\}$. After looking at $U \cap \{f = f(q)\}$, we may assume that the frontier of U in M traces a cylinder C tangent to X in the domain $\{f(q) - \varepsilon' \leq f \leq f(q) + \varepsilon'\}$, where $\varepsilon' > 0$ satisfies assumption ii) of our theorem. We then choose Morse charts $U(p)$ and $U(q)$ about p and q respectively, both contained in U . They are equipped with coordinates $(\tilde{y}, z) \in \mathbb{R}^{k+1} \times \mathbb{R}^{n-k-1}$ (resp. $(y, \tilde{z}) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$) such that $f|_{U(p)} = f(p) - |\tilde{y}|^2 + |z|^2$ (resp. $f|_{U(q)} = f(q) - |y|^2 + |\tilde{z}|^2$). Moreover, it is possible to require that $(\tilde{y}, 0)$ are coordinates of $W^u(p) \cap U(p)$ (thus, z must be coordinates in the orthogonal complement to $T_p W^u(p)$ with respect to the Hessian of f). Similarly, the coordinates of $U(q)$ may be chosen so that $(0, \tilde{z})$ are coordinates of $W^s(q) \cap U(q)$.

For $0 < \varepsilon < \varepsilon'$ small enough, there is a Morse model $\mathbb{M}(q) \subset U(q)$ disjoint from C whose top and bottom are respectively at levels $f(q) + \varepsilon$ and $f(q) - \varepsilon$; and also, there is a Morse model $\mathbb{M}(p) \subset U(p)$ with top and bottom at levels $f(p) + \varepsilon$ and $f(p) - \varepsilon$.

Denote by $\tilde{\ell}$ the trace of ℓ in the domain $\{f(q) + \varepsilon \leq f \leq f(p) - \varepsilon\}$. One end point a of $\tilde{\ell}$ is $W^u(p) \cap \{f = f(p) - \varepsilon\} \cap W^s(q)$ and the other end point b is $W^u(p) \cap \{f = f(q) + \varepsilon\} \cap W^s(q)$. Consider the diameter $\Delta_a \subset U(p)$ of the ball $\{|\tilde{y}|^2 \leq \varepsilon, z = 0\}$ ending at a . Split the coordinates (\tilde{y}, z) as (t, y, z) so that t is a coordinate of Δ_a and:

$$f|_{U(p)} = f(p) - t^2 - |y|^2 + |z|^2.$$

Similarly, split $(y, \tilde{z}) = (y, t, z)$ in $U(q)$ so that $b \in \Delta_b := \{y = 0, z = 0\}$ and

$$f|_{U(q)} = f(q) - |y|^2 + t^2 + |z|^2.$$

It is easy to find a C^∞ parametrized arc $\alpha : [0, 1] \rightarrow M$, $u \in [0, 1] \mapsto \alpha(u) \in M$, whose image is denoted by A , and a sequence $0 < u_0 < u'_0 < u'_1 < u_1 < 1$ with the following properties:

- $p = \alpha(u_0)$, $a = \alpha(u'_0)$, $b = \alpha(u'_1)$, $q = \alpha(u_1)$ and $\alpha([u'_0, u'_1]) = \tilde{\ell}$;
- the image $\alpha([0, u'_1])$ is contained in $W^u(p)$ and the image $\alpha([u'_1, 1])$ is contained in $W^s(q)$;
- the function h defined by $h := f \circ \alpha$ has two critical points, one maximum in p , one local minimum in q (both non-degenerate) and varies from $h(0) = f(q) - \varepsilon$ to $h(1) = f(q) + \varepsilon$.

One observes that there exists an increasing smooth function h_1 without critical points, which coincides with h near the extremities of $[0, 1]$, and satisfies:

$$h_1 \leq h. \tag{*}$$

This is the solution of the cancellation problem in the one-dimensional case.

We now construct a $(k + 1)$ -dimensional sub-manifold $W \subset U$ containing A such that p and q are the only critical points of $f|_W$. This W will contain the part of $W^u(p)$ above the level $f = f(q) + \varepsilon$, that is, $W^u(p) \cap \{f \geq f(q) + \varepsilon\} \cong D^{k+1}$. Let S be the boundary of this disc. Let T denote the top of $\mathbb{M}(q)$; we have $T \cong D^k \times S^{n-k-1}$ and the product structure of T is determined by the Morse coordinates. The point b belongs to the so-called *belt sphere* $\{0\} \times S^{n-k-1} \subset W^s(q)$. The radius of D^k measured with the norm of the y -coordinates is a free choice for the Morse model $\mathbb{M}(q)$. If this radius is small enough, the transversality assumption implies that $S \cap T$ is isotopic in T to the *standard meridian* $D^k \times \{b_0\}$ by an isotopy keeping the belt sphere fixed. Therefore, by a first level preserving isotopy of the coordinates of $\mathbb{M}(q)$ one makes $S \cap T$ be contained in $\{z = 0\}$. By a second isotopy of this type, one makes X and ∇f coincide near the top of $\mathbb{M}(q)$; here ∇f stands for the descending gradient of f in $\mathbb{M}(q)$ associated with the Euclidean metric of the coordinates (t, y, z) . These two isotopies leave $W^s(q)$ invariant and are stationary near q .

Let $B \subset \mathbb{M}(q)$ be defined by $B := \{z = 0\}$; it is C^∞ tangent to $W^u(p)$ along $B \cap S$. By partition of unity one finds a pseudo-gradient ξ such that:

- $\xi = X$ on $W^u(p)$ above the level $f = f(q) + \varepsilon$ and on the cylinder C defined previously;
- $\xi = \nabla f$ in $\mathbb{M}(q)$.

Let $W_\xi^u(p)$ be the unstable manifold of p for ξ . Each orbit of ξ in $W_\xi^u(p)$ that does not meet $\tilde{\ell}$ reaches the level set $\{f = f(q) - \varepsilon\}$ and remains in U since C prevents it from getting out. We define W as the union of B and $W_\xi^u(p)$ truncated by removing $\{f < f(q) - \varepsilon\}$.

Lemma 1. *There exist a tubular neighborhood N of A in U and coordinates $(u, y, z) \in [0, 1] \times \mathbb{R}^k \times \mathbb{R}^{n-k-1}$ of N , such that:*

- i) $A = \{y = 0, z = 0\}$;
- ii) $N_0 := N \cap W = \{z = 0\}$;
- iii) $f(u, y, z) = h(u) - |y|^2 + |z|^2$ at every point of N .

Proof. The first two items are easy to realize: take a provisory pair of small tubular neighborhoods (N, N_0) of A in (M, W) whose fibers are contained in $\{t = \text{const}\}$ with respect to the coordinates of $\mathbb{M}(p)$ and $\mathbb{M}(q)$. Choose the coordinates y in the fiber of N_0 over A and z in the fiber of N over N_0 such that (y, z) are the so-named coordinates in $\mathbb{M}(p)$ and $\mathbb{M}(q)$. For proving item iii), we need to invoke a general fact stated below.

Claim. *Let V be a smooth manifold and $V' \subset V$ be a sub-manifold. Two germs of smooth functions f and g along V' whose restrictions to V' coincide and have no critical points are isotopic relative to V' (meaning that they become equal after moving one of them by an isotopy of V fixing V'); moreover, if $f = g$ near a compact set $K \subset V'$, the isotopy may be the identity near K .*

For proving this claim, the path method of Moser is available; it may be applied to the path $t \in [0, 1] \mapsto f_t := (1 - t)f + tg$, that is: there is a time-depending vector field Z_t whose integration solves the problem of conjugating each f_t to f_0 near V' smoothly in $t \in [0, 1]$. It is sufficient to find Z_t satisfying $df_t(x).Z_t(x) = f(x) - g(x)$ near V' (see [8]); for that, apply the implicit function theorem for finding local solutions which are then glued by partition of unity. \square

We are going to apply this claim twice. First, apply it to the data $V = W, V' = A, K = A \cap (\mathbb{M}(p) \cup \mathbb{M}(q))$, the restriction $f|_{N_0}$ and $g(u, y) = h(u) - |y|^2$. The isotopy yielded by the claim moves the coordinates (u, y) , without changing u on A , so that the equality of iii) holds true in a neighborhood $N_1 \subset N_0$ of A in W . Then, apply the claim to the data: $V = M, V' = N_1, K = N_1 \cap (\mathbb{M}(p) \cup \mathbb{M}(q))$, f and $g(u, y, z) = h(u) - |y|^2 + |z|^2$. The new isotopy finishes to put the coordinates in a position that makes the equality of item iii) hold true near A in M , yielding the desired N . \square

In $W \cap N$, the vector field $Y := y\partial_y$ is tangent to W and it is a pseudo-gradient for $f|(W \cap (N \setminus A))$. Still denoted Y , it extends to the whole W as a pseudo-gradient for $f|(W \setminus A)$ (by partition of unity). Each orbit of Y converges to a point of A in the past. This defines a k -disk fibration $W \rightarrow A$ which is pinched at $\alpha(0) \in A$. Indeed, this fibration is clear for $W \cap N \rightarrow A$ and the orbits of Y yield an ambient isotopy from W to $W \cap N$. Denote D_u the fiber of W over $\alpha(u)$.

Let \tilde{W} be a tubular neighborhood of W in $\{f \geq f(q) - \varepsilon\}$. The fibration $W \rightarrow A$ extends to \tilde{W} as an $(n - 1)$ -disk fibration:

$$\tilde{D}_u \hookrightarrow \tilde{W} \rightarrow A,$$

pinched at $\alpha(0)$ and coinciding with the projection $(u, y, z) \rightarrow \alpha(u)$ in N . The function $f_u := f|_{\tilde{D}_u}$ is a Morse function with $\alpha(u)$ as unique critical point and its index is k . It has a pseudo-gradient \tilde{Y}_u , tangent to \tilde{D}_u , which coincides with Y on W and with $y\partial_y - z\partial_z$ on N .

Lemma 2 (Decreasing of a critical value). *Let $g : V \rightarrow \mathbb{R}$ be a Morse function with a pseudo-gradient Z . Let p be a critical point of index k and let $a < g(p)$. Assume that the unstable manifold $W^u(p)$ contains a compact k -disk D whose boundary lies in the level set $\{g = a\}$. Let U be a neighborhood of D in $\{g \geq a\}$. Then, for every $\varepsilon > 0$, there exists a one parameter family of Morse functions $(g_t)_{t \in [0, 1]}$, such that:*

- $g_0 = g, a < g_1(p) < a + \varepsilon,$
- Z is a pseudo-gradient of g_t for every $t \in [0, 1],$
- and $g_t = g$ out of $U.$

The same statement holds true with parameters.

Proof. The point p is a hyperbolic zero of the vector field Z . Then, the orbits of Z close to D but not tangent to D are crossing the level set $\{g = g(p) + \eta\}$, for a small $\eta > 0$ and, of course, the level set $\{g = a\}$. Consider the foliation \mathcal{F} defined by $g = \text{const}$ on $U \setminus D$. Find an n -dimensional compact domain K in $U \setminus D$ made of pieces of Z -orbits from $\{g = g(p) + \eta\}$ to $\{g = a\}$ surrounding D . It is easy to replace $\mathcal{F}|_K$ by a foliation whose leaves are still transverse to Z in order to have the level sets of the wanted function g_1 . We refer to [3] for more details (see also [5] Section 4 or [9] Section 2). \square

End of the proof of the theorem. The critical point of f_u is $p_u := \alpha(u) = (u, 0, 0)$ and its value is $h(u)$. The vector field \tilde{Y}_u is a pseudo-gradient for f_u and the disk D_u fulfills the requirement of Lemma 2, which we apply smoothly with respect to the parameter u . So, it is possible to decrease the critical value of f_u , smoothly in $u \in [0, 1]$, from $h(u)$ to $h_1(u)$, still keeping \tilde{Y}_u as a pseudo-gradient. Therefore, the deformation of f over A from h to h_1 (compare $(*)$) extends to \tilde{W} without creating new critical points away from A . Moreover, the deformation is supported in \tilde{W} , hence in U , as wanted. \square

3. Applications

In this section, two more classical statements are given starting with the one from Milnor's book ([5], Section 5).

Corollary 1. *Let (W, L_0, L_1) be a compact cobordism and $f : (W, L_0, L_1) \rightarrow ([0, 1], 0, 1)$ be a Morse function with two critical points p and q . Assume there is a pseudo-gradient such that $W^u(p)$ and $W^s(q)$ intersect in one orbit and transversely. Then the cobordism is a product: $W \cong M_0 \times [0, 1].$*

Proof. Certainly $0 < f(q) < f(p) < 1$. By compactness, every orbit of the pseudo-gradient in $W^u(p)$ distinct from the connecting orbit reaches L_0 . Hence, the assumptions of our theorem are fulfilled. After canceling the critical points, W is a product by gradient lines. \square

Corollary 2. *Let M be a closed manifold, $f : M \rightarrow \mathbb{R}$ be a Morse function, (p, q) be a pair of critical points whose respective indices are $k + 1$ and k . Let X be a pseudo-gradient of f . Assume that there are compact disks $D(p) \subset W^u(p)$ and $D(q) \subset W^s(q)$ with the following properties:*

- $D(p)$ and $D(q)$ are neighborhoods of p and q respectively in $W^u(p)$ and $W^s(q)$;
- their boundaries lie in a regular level set $\{f = a\}$, $f(q) < a < f(p)$, in which they intersect in one point only and transversely.

Then, the pair (p, q) is cancelable.

Proof. Since a is a regular value, the disk $D(q)$ can be extended (keeping its name), so that its boundary lies in $\{f = a + \varepsilon\}$, for some small $\varepsilon > 0$. Lemma 2 can be applied for decreasing the critical value $f(p)$ so that $a < f(p) < a + \varepsilon$. Similarly, by considering $-f$, Lemma 2 can be applied for increasing the critical value $f(q)$ so that $a < f(q) < f(p) < a + \varepsilon$. In this situation, as in Corollary 1, the assumptions of our theorem are fulfilled. The cancellation follows. \square

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