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Band structure of the Ruelle spectrum of contact Anosov flows

Structure en bandes du spectre de Ruelle des flots d'Anosov de contact

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ABSTRACT

If X is a contact Anosov vector field on a smooth compact manifold M and $V \in C^\infty(M)$, it is known that the differential operator $A = -X + V$ has some discrete spectrum called Ruelle–Pollicott resonances in specific Sobolev spaces. We show that for $|\mathrm{Im}\, z| \to \infty$ the eigenvalues of A are restricted to vertical bands and in the gaps between the bands, the resolvent of A is bounded uniformly with respect to $|\mathrm{Im}(z)|$. In each isolated band, the density of eigenvalues is given by the Weyl law. In the first band, most of the eigenvalues concentrate to the vertical line $\mathrm{Re}(z) = \langle D \rangle_M$, the space average of the function $D(x) = V(x) - \frac{1}{2} \operatorname{div} X_{|E_u(x)}$ where E_u is the unstable distribution. This band spectrum gives an asymptotic expansion for dynamical correlation functions.

RÉSUMÉ

Si X est un champ de vecteur d'Anosov de contact sur une variété compacte lisse M et si $V \in C^\infty(M)$, il est connu que l'opérateur différentiel $A = -X + V$ a un spectre discret appelé résonances de Ruelle–Pollicott dans des espaces de Sobolev spécifiques. On montre que, pour $|\mathrm{Im}\, z| \to \infty$, les valeurs propres de A sont incluses dans des bandes verticales et que, dans les gaps entre ces bandes, la résolvante de A est bornée uniformément par rapport à $|\mathrm{Im}(z)|$. Dans chaque bande isolée, la densité des valeurs propres est donnée par une loi de Weyl. Dans la première bande, la plupart des valeurs propres se concentrent sur la ligne verticale $\mathrm{Re}(z) = \langle D \rangle_M$, qui est la moyenne spatiale de la fonction $D(x) = V(x) - \frac{1}{2} \operatorname{div} X_{|E_u(x)}$, où E_u est la distribution instable. Ce spectre en bande permet d'exprimer le comportement asymptotique des fonctions de corrélations dynamiques.

1. Introduction

In this paper, we announce some results concerning the Ruelle–Pollicott spectrum of transfer operators associated with contact Anosov flows [8]. Let X be a smooth vector field on a compact manifold M and suppose that X generates a contact Anosov flow.

The Ruelle–Pollicott spectrum of contact Anosov flows has been studied since a long time due to its importance to describe the precise behavior and decay of time correlation functions for a large time. From this, one can deduce fine

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statistical properties of the dynamics of the flow such as the exponential convergence towards equilibrium (i.e. mixing) or the central limit theorem for the Birkhoff average of functions. The Ruelle–Pollicott spectrum is also useful to get some precise asymptotic counting of periodic orbits.

Recent results show that the Ruelle–Pollicott resonances are the discrete eigenvalues of the generator $(-X)$ seen as a differential operator in some specific Sobolev spaces of distributions $\mathcal{H} \subset \mathcal{D}'(M)$ [1,6,9]. A more precise description of the structure of this spectrum has been obtained in [15,16], where it is shown that in the asymptotic limit $|\mathrm{Im}\, z| \to \infty$ the spectrum is on the domain $\mathrm{Re}(z) \leqslant \gamma_0^+$ with some explicit "gap" $\gamma_0^+ < 0$ given below. More generally, these results can be extended to the operator $A = -X + V$, where $V \in C^\infty(M)$ is a smooth function called "potential".

In this paper, we improve the description of the structure of this Ruelle–Pollicott spectrum. The main results are stated in Theorem 5.1. They show that the Ruelle–Pollicott spectrum of the first-order differential operator $A = -X + V$ has some band structure in the asymptotic limit $|\mathrm{Im}\, z| \to \infty$, i.e. is contained in the union of vertical bands $\mathbf{B}_k = \{z \in \mathbb{C},\ \mathrm{Re}(z) \in [\gamma_k^-, \gamma_k^+]\}$, $k \geqslant 0$ with $\gamma_{k+1}^\pm < \gamma_k^\pm$. The values γ_k^+, γ_k^- are given explicitly in (5.1) by the maximum (respect. minimum) of the time averaged along trajectories of a function $D \in C^\infty(M)$ called "damping function" given by $D = V - \frac{1}{2} \operatorname{div} X_{|E_u}$. If the band $\mathbf{B}_k$ is isolated from the others by an asymptotic spectral gap (i.e. $\gamma_{k+1}^+ < \gamma_k^-$), then the norm of the resolvent of A is bounded in this gap uniformly with respect to $|\mathrm{Im}(z)|$. Theorem 5.1 shows that the spectrum in every isolated band $\mathbf{B}_k$ satisfies a Weyl law, i.e. the number $\mathcal{N}(b)$ of eigenvalues $z \in \mathbf{B}_k$ satisfying $\mathrm{Im}(z) \in [b, b + b^\varepsilon]$ is given by[1]: $\mathcal{N}(b)/b^\varepsilon \asymp b^d$ as $b \to \infty$ for any $\varepsilon > 0$, where $\dim M = 2d + 1$. The assumption that the band is isolated is not needed for the upper bound. A better result for the upper bound of this Weyl law is given in [2]: it is shown that for any radius $C_0 > 0$, the number of resonances in the disk $D(ib, C_0)$ of center ib is $O(b^d)$ (i.e. this is the case $\varepsilon = 0$).

Concerning the most interesting "external band" $\mathbf{B}_0 = \{z \in \mathbb{C},\ \mathrm{Re}(z) \in [\gamma_0^-, \gamma_0^+]\}$, supposing that it is isolated $(\gamma_1^+ < \gamma_0^-)$, it is shown in Theorem 5.3 that most of the resonances in the band $\mathbf{B}_0$ accumulate on the vertical line $\mathrm{Re}(z) = \langle D \rangle_M$ given by the space average of the function D. This is due to ergodicity. This problem is then closely related to the description of the spectrum of the damped-wave equation [13]. Finally, Corollary 5.4 shows that dynamical correlation functions can be expanded over the infinite spectrum contained in the first band $\mathbf{B}_0$.

In the forthcoming paper [8], we will consider the special case $V = V_0 = \frac{1}{2} \operatorname{div} X_{|E_u}$ for which the damping function vanishes $D = 0$, $\gamma_0^\pm = 0$, i.e. the Ruelle–Pollicott resonances of the external band accumulate on the imaginary axis. However, V_0 is not smooth and this requires an extension of the theory.

From the Selberg theory and the representation theory, this particular band structure is known for a long time in the case of homogeneous hyperbolic manifolds $M = \Gamma \backslash SO_{1,n}/SO_{n-1} \equiv \Gamma \backslash T_1^* \mathbb{H}^n$, where Γ is a discrete co-compact subgroup. In that case, the contact Anosov flow is the geodesic flow on the hyperbolic manifold surface $\mathcal{N} = \Gamma \backslash \mathbb{H}^n = \Gamma \backslash SO_{1,n}/SO_n$.

Technically, we use a semiclassical analysis to study the spectrum of the differential operator $A = -X + V$ [11,17]. We consider the associated "canonical dynamics" in the phase space T^*M, which is simply the lifted flow. The key observation is that this canonical dynamics has a non-wandering set or "trapped set", which is a smooth symplectic submanifold $K \subset T^*M$ and which is normally hyperbolic. This is the origin of the band structure of the spectrum. The results presented in this paper have already been obtained (among others) for a closely related problem, namely the band structure of prequantum Anosov diffeomorphisms [7]. This approach has been originally developed on a simple model in [5].

In a recent paper [4], Semyon Dyatlov shows a band structure for resonances for a similar problem motivated by scattering by black holes. The band structure he obtains also comes from the property that the trapped set in his problem is symplectic and normally hyperbolic, but he assumes some smoothness for the (un)stable foliations. One difficulty we have to deal with for Anosov flows is the non-smoothness of the (un)stable foliations. Another related recent work is the paper of Nonnenmacher and Zworski [12], where they obtain Theorem 3.5 below, but for more general models, including the contact Anosov flow.

2. Contact Anosov flow

Definition 2.1. On a smooth Riemannian compact manifold (M, g), a smooth vector field X generating a flow $\phi_t : M \to M$, $t \in \mathbb{R}$, is **Anosov** (see Fig. 2.1) if there exists a ϕ_t-invariant decomposition of the tangent bundle $TM = E_u \oplus E_s \oplus E_0$, where $E_0 = \mathbb{R}X$ and $C > 0$, $\lambda > 0$ such that for every $t \geqslant 0$:

$$\|D\phi_{t/E_s}\|_g \leqslant C e^{-\lambda t}, \qquad \|D\phi_{-t/E_u}\|_g \leqslant C e^{-\lambda t}. \tag{2.1}$$

Remark 2.2. In general, the map $x \in M \to E_u(x), E_s(x)$ is only Hölder continuous. The "structural stability theorem" shows that the Anosov vector field is a property robust under perturbation.

Definition 2.3. The Anosov one-form $\alpha \in C(T^*M)$ is defined by $\mathrm{Ker}\,\alpha = E_u \oplus E_s$, $\alpha(X) = 1$. X is a **contact Anosov vector field** if α is a smooth contact one-form, i.e. $(d\alpha)_{|E_u \oplus E_s}$ is non-degenerate (symplectic).

[1] The notation $\mathcal{N}(b) \asymp |b|^{d+\varepsilon}$ means that $\exists C > 0$ independent of b s.t. $\frac{1}{C}|b|^{d+\varepsilon} \leqslant \mathcal{N}(b) \leqslant C|b|^{d+\varepsilon}$.

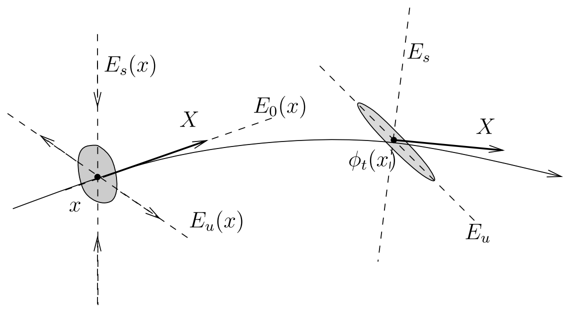

Fig. 2.1. Anosov flow.

Remark 2.4. In the case of a contact Anosov vector field, we have: $\dim E_u = \dim E_s = d$, with $\dim M = 2d+1$ and that $dx = \alpha \wedge (d\alpha)^d$ is smooth volume form on M preserved by the flow ϕ_t.

As an example, the geodesic flow on a compact manifold $\mathcal{N}$ with negative sectional curvature (not necessary constant) defines a contact Anosov flow on $T_1^*\mathcal{N}$. In that case, the Anosov one-form α coincides with the canonical Liouville one-form ξdx on $T^*\mathcal{N}$.

We will assume that X is a contact Anosov vector field on M in the rest of this paper.

3. The transfer operator

Let $V \in C^\infty(M)$ be a smooth function called "potential".

Definition 3.1. The **transfer operator** is the group of operators:

$$\hat{F}_t\colon \begin{cases} C^\infty(M) & \to C^\infty(M) \\ v & \to e^{tA} v \end{cases}, \quad t \geqslant 0$$

with the generator:

$$A := -X + V$$

which is a first-order differential operator.

Remark 3.2.

- Since X generates the flow ϕ_t we can write $\hat{F}_t v = (\mathrm{e}^{\int_0^t V\circ\phi_{-s}\,\mathrm{d}s}) v(\phi_{-t}(x))$, hence $\hat{F}_t$ acts as transport of functions by the flow with multiplication by exponential of the function V averaged along the trajectory.
- In the case $V = 0$, the operator $\hat{F}_t$ is useful in order to express "dynamical correlation functions" between $u, v \in C^\infty(M)$, $t \in \mathbb{R}$:

$$C_{u,v}(t) := \int_M u \cdot (v \circ \phi_{-t})\,\mathrm{d}x = \langle u, \hat{F}_t v\rangle_{L^2}. \tag{3.1}$$

The study of these time correlation functions permits to establish the mixing properties and other statistical properties of the dynamics of the Anosov flow. In particular, $u = cste$ is an obvious eigenfunction of $A = -X$ with eigenvalue $z_0 = 0$. Since $\operatorname{div} X = 0$, we have that $\hat{F}_t$ is unitary in $L^2(M, dx)$ and $iA = (iA)^*$ is self-adjoint and has a continuous spectrum on the imaginary axis $\operatorname{Re} z = 0$. In the next theorem, we consider more interesting functional spaces where the operator A has a discrete spectrum, but is non-self-adjoint.

Theorem 3.3 *(Discrete spectrum). (See [1,6].) If X is an Anosov vector field and $V \in C^\infty(M)$ then for every $C > 0$, there exists a Hilbert space $\mathcal{H}_C$ with $C^\infty(M) \subset \mathcal{H}_C \subset \mathcal{D}'(M)$, such that:*

$$A = -X + V : \mathcal{H}_C \to \mathcal{H}_C$$

*has a discrete spectrum on the domain $\operatorname{Re}(z) > -C\lambda$, called **Ruelle–Pollicott resonances**, independent of the choice of $\mathcal{H}_C$.*

Remark 3.4. Concerning the meaning of these eigenvalues, notice that with the choice $V = 0$, if $(-X)v = zv$, v is an invariant distribution with eigenvalue $z = -a + ib \in \mathbb{C}$, then $v \circ \phi_{-t} = \mathrm{e}^{-tX} v = \mathrm{e}^{-at}\mathrm{e}^{ibt} v$, i.e. $a = -\operatorname{Re}(z)$ contributes as a damping factor and $b = \operatorname{Im}(z)$ as a frequency in time correlation function (3.1). See Corollary 5.4 below for a precise statement. Notice also the symmetry of the spectrum under complex conjugation that $Av = zv$ implies $A\bar{v} = \bar{z}\bar{v}$.

We introduce now the following function that will play an important role[2]:

$$V_0(x) := \frac{1}{2}\operatorname{div} X_{|E_u}. \tag{3.2}$$

From (2.1), we have $V_0(x) \geqslant \frac{1}{2} d \cdot \lambda$. Since $E_u(x)$ is only Hölder in x, so is $V_0(x)$. We will also consider the difference:

$$D(x) := V(x) - V_0(x) \tag{3.3}$$

and call it the "**effective damping function**". For simplicity, we will write:

$$\left(\int_0^t D\right)(x) := \int_0^t (D \circ \phi_{-s})(x)\, \mathrm{d}s, \quad x \in M,$$

for the Birkhoff average of D along trajectories.

Theorem 3.5 *(Asymptotic gap). (See [15,16].) If X is a contact Anosov vector field on M and $V \in C^\infty(M)$, then for any $\varepsilon > 0$ the Ruelle–Pollicott eigenvalues $(z_j)_j \in \mathbb{C}$ of $A = -X + V$ are contained in*:

$$\operatorname{Re}(z) \leqslant \gamma_0^+ + \varepsilon$$

up to finitely many exceptions and with:

$$\gamma_0^+ = \lim_{t\to\infty} \sup_{x\in M} \frac{1}{t}\left(\int_0^t D\right)(x). \tag{3.4}$$

Remark 3.6. See Fig. 4.1(b). Notice that in the case $V = 0$, we have $\gamma_0^+ \leqslant -\frac{1}{2} d \cdot \lambda < 0$.

4. Example of the geodesic flow on a constant curvature surface

A simple and well-known example of the contact Anosov flow is provided by the geodesic flow on a surface $\mathcal{S}$ with a constant negative curvature. Precisely, let $\Gamma < SL_2\mathbb{R}$ be a co-compact discrete subgroup of $G = SL_2\mathbb{R}$ (i.e. such that $M := \Gamma\backslash SL_2\mathbb{R}$ is compact). We suppose that $(-\mathrm{Id}) \in \Gamma$. Then we have a natural identification that $M \equiv T_1^*\mathcal{S}$ is the unit cotangent bundle of the hyperbolic surface $\mathcal{S} := \Gamma\backslash SL_2\mathbb{R}/SO_2 \equiv \Gamma\backslash\mathbb{H}^2$. Let X be the left invariant vector field on M given by the element $X_e = \frac{1}{2}\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right) \in sl_2\mathbb{R} = T_e G$. Then X is an Anosov contact vector field on M and can be interpreted as the geodesic flow on the surface $\mathcal{S}$. Using the representation theory, it is known that the Ruelle–Pollicott spectrum of the operator $(-X)$ coincides with the zeros of the dynamical Fredholm determinant. This dynamical Fredholm determinant is expressed as the product of the Selberg zeta functions and gives the following result; see Fig. 4.1(a). We refer the reader to [8] for further details.

Proposition 4.1. *If X is the geodesic flow on an hyperbolic surface $\mathcal{S} = \Gamma\backslash\mathbb{H}^2$, then the Ruelle–Pollicott eigenvalues z of $(-X)$ are of the form*:

$$z_{k,l} = -\frac{1}{2} - k \pm i\sqrt{\mu_l - \frac{1}{4}} \tag{4.1}$$

where $k \in \mathbb{N}$ and $(\mu_l)_{l\in\mathbb{N}} \in \mathbb{R}^+$ are the discrete eigenvalues of the hyperbolic Laplacian Δ on the surface $\mathcal{S}$. There are also $z_n = -n$ with $n \in \mathbb{N}^*$. Each set $(z_{k,l})_l$ with fixed k will be called the line $\mathbf{B}_k$. The "Weyl law" for Δ gives the density of eigenvalues on each vertical line $\mathbf{B}_k$, for $b \to \infty$,

$$\sharp\left\{z_{k,l},\ b < \operatorname{Im}(z_{k,l}) < b+1\right\} \asymp |b|. \tag{4.2}$$

[2] Let μ_g be the induced Riemann volume form on $E_u(x)$ defined from the choice of a metric g on M. As the usual definition in differential geometry [14, p. 125], for tangent vectors $u_1, \ldots, u_d \in E_u(x)$, $\operatorname{div} X_{|E_u}$ measures the rate of change of the volume of E_u and is defined by:

$$\left(\operatorname{div} X_{|E_u}(x)\right) \cdot \mu_g(u_1, \ldots, u_d) = \lim_{t\to 0} \frac{1}{t}\left(\mu_g\left(D\phi_t(u_1), \ldots, D\phi_t(u_d)\right) - \mu_g(u_1, \ldots, u_d)\right).$$

Equivalently we can write that $\operatorname{div} X_{|E_u}(x) = \frac{\mathrm{d}}{\mathrm{d}t}(\det(D\phi_t)_{|E_u})_{t=0}$.

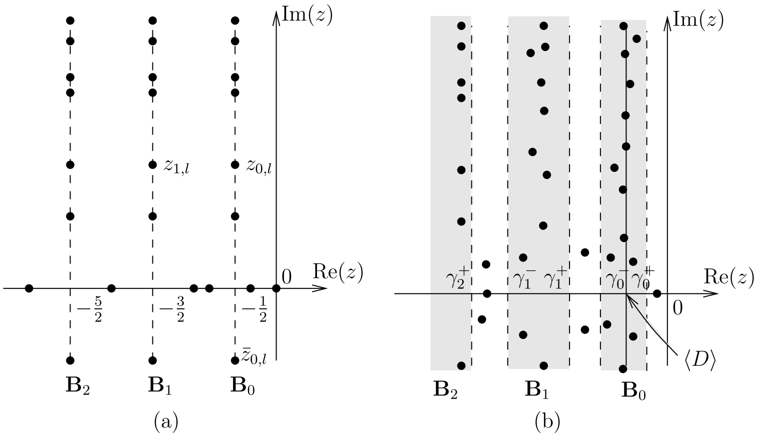

Fig. 4.1. (a) For a hyperbolic surface $\mathcal{S} = \Gamma\backslash\mathbb{H}^2$, the Ruelle–Pollicott spectrum of the geodesic vector field $-X$ is given by Proposition 4.1. It is related to the eigenvalues of the Laplacian by (4.1). (b) For a general contact Anosov flow, the spectrum of $A = -X + V$ and its asymptotic band structure is given by Theorems 5.1 and 5.3.

5. Band spectrum for the general contact Anosov flow

Proposition 4.1 above shows that the Ruelle–Pollicott spectrum for the geodesic flow on a constant negative surface has the structure of vertical lines $\mathbf{B}_k$ at $\operatorname{Re} z = -\frac{1}{2} - k$. In each line, the eigenvalues are in correspondence with the eigenvalues of the Laplacian Δ. We address now the question if this structure persists somehow for a geodesic flow on manifolds with negative (variable) sectional curvature and, more generally, for any contact Anosov flow. In the next theorem, for a linear invertible map L, we note $\|L\|_{\max} := \|L\|$ and $\|L\|_{\min} := \|L^{-1}\|^{-1}$.

Theorem 5.1 *(Asymptotic band structure). (See [8].) If X is a contact Anosov vector field on M and $V \in C^\infty(M)$, then for every $C > 0$, there exists a Hilbert space $\mathcal{H}_C$ with $C^\infty(M) \subset \mathcal{H}_C \subset \mathcal{D}'(M)$, such that for any $\varepsilon > 0$, the Ruelle–Pollicott eigenvalues $(z_j)_j \in \mathbb{C}$ of the operator $A = -X + V : \mathcal{H}_C \to \mathcal{H}_C$ on the domain $\operatorname{Re}(z) > -C\lambda$ are contained, up to finitely many exceptions, in the union of finitely many bands*:

$$z \in \bigcup_{k\geqslant 0} \underbrace{\left[\gamma_k^- - \varepsilon, \gamma_k^+ + \varepsilon\right] \times i\mathbb{R}}_{\text{Band } \mathbf{B}_k}$$

with

$$\gamma_k^+ = \lim_{t\to\infty} \left| \sup_x \frac{1}{t} \left(\left(\int_0^t D \right)(x) - k \log \left\| D\phi_t(x)_{/E_u} \right\|_{\min} \right) \right|, \tag{5.1}$$

$$\gamma_k^- = \lim_{t\to\infty} \left| \inf_x \frac{1}{t} \left(\left(\int_0^t D \right)(x) - k \log \left\| D\phi_t(x)_{/E_u} \right\|_{\max} \right) \right| \tag{5.2}$$

and where $D = V - V_0$ is the damping function (3.3). In the gaps (i.e. between the bands), the norm of the resolvent is controlled: there exists $c > 0$ such that, for every $z \notin \bigcup_{k\geqslant 0} \mathbf{B}_k$ with $|\operatorname{Im}(z)| > c$:

$$\left\| (z - A)^{-1} \right\| \leqslant c.$$

*For some $k \geqslant 0$, if the band $\mathbf{B}_k$ is "isolated", i.e. $\gamma_{k+1}^+ < \gamma_k^-$ and $\gamma_k^+ < \gamma_{k-1}^-$ (this last condition is for $k \geqslant 1$) then the number of resonances in $\mathbf{B}_k$ obeys a "**Weyl law**": $\forall b > c$,*

$$\frac{1}{c}|b|^d < \frac{1}{|b|^\varepsilon} \cdot \sharp\left\{ z_j \in \mathbf{B}_k, b < \operatorname{Im}(z_j) < b + b^\varepsilon \right\} < c|b|^d \tag{5.3}$$

with $\dim M = 2d + 1$. The upper bound holds without the condition that $\mathbf{B}_k$ is isolated.

Remark 5.2. We can compare Theorem 5.1 with Proposition 4.1 in the special case of the geodesic flow on a constant curvature surface $\mathcal{S} = \Gamma\backslash\mathbb{H}^2$: we have $D\phi_t(x)_{/E_u} \equiv e^t$, hence $V_0 = \frac{1}{2}$. The choice of potential $V = 0$ gives the constant damping function $D = -\frac{1}{2}$, hence (5.1) gives $\gamma_k^+ = \gamma_k^- = -\frac{1}{2} - k$ as in Proposition 4.1.

In the forthcoming paper [8], we will show that, for a general contact Anosov vector field, it is possible to choose the potential $V = V_0$ (non-smooth), giving $\gamma_0^+ = \gamma_0^- = 0$, i.e. the first band is reduced to the imaginary axis and is isolated from the second band by a gap, $\gamma_1^+ < 0$.

Theorem 5.3. *(See [8].) If the external band* $\mathbf{B}_0$ *is isolated, i.e.* $\gamma_1^+ < \gamma_0^-$*, then most of the resonances accumulate on the vertical line*:

$$\mathrm{Re}(z) = \langle D\rangle := \frac{1}{\mathrm{Vol}(M)}\int_M D(x)\,\mathrm{d}x$$

in the precise sense that:

$$\frac{1}{N_b}\sum_{z_i\in\mathbf{B}_0,\,|\mathrm{Im}(z_i)|<b}\big|z_i-\langle D\rangle\big|\underset{b\to\infty}{\longrightarrow}0,\quad \text{with } N_b := \sharp\big\{z_i\in\mathbf{B}_0,\ \big|\mathrm{Im}(z_i)\big|<b\big\}. \tag{5.4}$$

5.1. Consequence for correlation functions' expansion

We mentioned the usefulness of dynamical correlation functions in (3.1). Let Π_j denote the finite rank spectral projector associated with the eigenvalue z_j. The following corollary provides an expansion of the correlation functions over the spectrum of resonances of the first band $\mathbf{B}_0$. This is an infinite sum.

Corollary 5.4. *Suppose that* $\gamma_1^+ < \gamma_0^-$*. Then for any* $\varepsilon > 0$*, there exists* C_ε*, for any* $u, v \in C^\infty(M)$ *and* $t \geqslant 0$*,*

$$\left|\langle u, \hat{F}_t v\rangle_{L^2} - \sum_{z_j,\mathrm{Re}(z_j)\geqslant\gamma_1^++\varepsilon}\langle u, \hat{F}_t\Pi_j v\rangle\right| \leqslant C_\varepsilon\|u\|_{\mathcal{H}'_C}\cdot\|u\|_{\mathcal{H}_C}\cdot e^{(\gamma_1^++\varepsilon)t}. \tag{5.5}$$

The infinite sum above converges fast because for arbitrary large $m \geqslant 0$ *there exists* $C_{m,\varepsilon}(u,v) \geqslant 0$ *such that* $|\langle u, \hat{F}_t\Pi_j v\rangle| \leqslant C_{m,\varepsilon}(u,v)\cdot e^{(\gamma_0^++\varepsilon)t}\cdot|\mathrm{Im}(z_j)|^{-m}$ *(except for a finite number of terms).*

Eq. (5.5) is a refinement of decay of correlation results of Dolgopyat [3], Liverani [10], Tsujii [15], [16, Corollary 1.2] and Nonnenmacher and Zworski [12, Corollary 5], where their expansion is a finite sum over one or a finite number of leading resonances.

5.2. Outline of the proof

The band structure and all related results presented above have already been proven for the spectrum of Anosov prequantum map in [7]. An Anosov prequantum map $\tilde{f}: P \to P$ is an equivariant lift of an Anosov diffeomorphism $f: M \to M$ on a principal bundle $U(1) \to P \to M$ such that $\tilde{f}$ preserves a contact one-form α (a connection on P). Therefore, $\tilde{f}: P \to P$ is very similar to the contact Anosov flow $\phi_t: M \to M$ considered in this paper, which also preserves a contact one-form α. Our proof of Theorem 5.1 is directly adapted from the proof given in [7]. We refer the reader to this paper for more precisions on the proof and we use the same notations below. The techniques rely on a semiclassical analysis adapted to the geometry of the contact Anosov flow lifted in the cotangent space T^*M. In the limit $|\mathrm{Im}\,z| \to \infty$ of large frequencies under study, the semiclassical parameter is written $\hbar := 1/|\mathrm{Im}\,z|$. We now sketch the main steps of the proof.

Global geometrical description. $A = -X + V$ is a differential operator. Its principal symbol is the function $\sigma(A)(x,\xi) = X_x(\xi)$ on the phase space T^*M (the cotangent bundle). It generates a Hamiltonian flow, which is simply the canonical lift of the flow ϕ_t on M. Due to Anosov's hypothesis on the flow in Definition 2.1, the non-wandering set of the Hamiltonian flow is the continuous sub-bundle $K = \mathbb{R}\alpha \subset T^*M$, where α is the Anosov one-form. K is normally hyperbolic. This analysis has already been used in [6] for the semiclassical analysis of Anosov flow (not necessary contact). With the additional hypothesis that α is a smooth contact one-form, this makes $K\backslash\{0\}$ a smooth symplectic submanifold of T^*M (usually called the symplectization of the contact one-form α) and normally hyperbolic. Let $\rho = (x,\xi) \in K$ be a point on the trapped set. Let $\hbar^{-1} = X_x(\xi)$ be its "energy". Let $\Omega = \sum_j dx^j \wedge d\xi^j$ be the canonical symplectic form on T^*M and consider the Ω-orthogonal splitting of the tangent space at $\rho \in K$:

$$T_\rho\big(T^*M\big) = T_\rho K \oplus (T_\rho K)^\perp. \tag{5.6}$$

Due to the hyperbolicity assumption, we have an additional decomposition of the space:

$$(T_\rho K)^\perp = E_u^{(2)} \oplus E_s^{(2)}$$

transverse to the trapped set into unstable/stable spaces.

Partition of unity. We decompose functions on the manifold using a microlocal partition of unity of size $\hbar^{1/2-\varepsilon}$ with some $1/2 > \varepsilon > 0$, that is refined as $\hbar \to 0$. In each chart, we use a canonical change of variables adapted to the decomposition (5.6), and construct an escape function adapted to the local splitting $E_u^{(2)} \oplus E_s^{(2)}$ above. This escape function has a "strong damping effect" outside a vicinity of size $O(\hbar^{1/2})$ of the trapped set K. We use this to define the anisotropic Sobolev space $\mathcal{H}_C$. At the level of operators, we perform a decomposition similar to that in (5.6) and obtain a microlocal decomposition of the transfer operator $\hat{F}_t$ as a tensor product $\hat{F}_{t|T_\rho K} \otimes \hat{F}_{t|(T_\rho K)^\perp}$. The first operator $\hat{F}_{t|T_\rho K}$ is unitary, whereas the second one $\hat{F}_{t|(T_\rho K)^\perp}$ has a discrete spectrum of resonances indexed by an integer $k \in \mathbb{N}$. This is due to the choice of the escape function. We can construct explicitly some approximate local spectral projectors Π_k for every value of k, and patching these local expressions together we get global spectral operators for each band. The positions $\gamma_k^\pm$ of the band $\mathbf{B}_k$ come from estimates on the discrete spectrum of the local operator $\hat{F}_{t|(T_\rho K)^\perp}$ restricted by the projector Π_k. We obtain results on the spectrum of the generator A from the results on the transfer operator $\hat{F}_t = e^{tA}$ by standard arguments.

The proof of the Weyl law is similar to the proof of J. Sjöstrand about the damped-wave equation [13], but needs more arguments. The accumulation of resonances on the value $\langle D\rangle$ in Theorem 5.3 given by the spatial average of the damping function, Eq. (5.4), uses the ergodicity property of the Anosov flow and is also similar to the spectral results obtained in [13] for the damped-wave equation.

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