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ABSTRACT

The conformal module of conjugacy classes of braids appeared in a paper of Lin and Gorin in connection with their interest in the 13th Hilbert Problem. This invariant is the supremum of conformal modules (in the sense of Ahlfors) of certain annuli related to the conjugacy class. This note states that the conformal module is inversely proportional to a popular dynamical braid invariant, the entropy. The entropy appeared in connection with Thurston's theory of surface homeomorphisms. An application of the concept of conformal module to algebraic geometry is given.

RÉSUMÉ

Le module conforme des classes de conjugaison de tresses est apparu dans un article de Lin et Gorin dans le cadre de leur intérêt pour le 13^e problème de Hilbert. Cet invariant est la borne supérieure des modules conformes (dans le sens d'Ahlfors) de certains anneaux associés à la classe de conjugaison. Cette note affirme que le module conforme est inversement proportionnel à un invariant dynamique bien connu des tresses, l'entropie. L'entropie est apparue dans le cadre de la théorie de Thurston des homéomorphismes de surfaces. Une application du concept de module conforme à la géométrie algébrique est donnée.

Braids occur in several mathematical fields, sometimes unexpectedly, and one can think of them in several different ways. Braids on n strands can be interpreted as algebraic objects, namely, as elements of the Artin group $\mathcal{B}_n$, or as isotopy classes of geometric braids, or as elements of the mapping class group of the n-punctured disc [5]. In connection with his interest in the Thirteen's Hilbert Problem, Arnol'd gave the following interpretation of the braid group $\mathcal{B}_n$. Denote by $\mathfrak{P}_n$ the space of monic polynomials of degree n without multiple zeros. This space can be parameterized either by the coefficients or by the unordered tuple of zeros of polynomials. This makes $\mathfrak{P}_n$ a complex manifold, in fact, the complement of the algebraic hypersurface $\{D_n = 0\}$ in the complex Euclidean space $\mathbb{C}^n$. Here $D_n(p)$ denotes the discriminant of the polynomial p. The function D_n is a polynomial in the coefficients of p. Arnol'd studied the topological invariants of $\mathfrak{P}_n$ [2]. Choose a base point $p \in \mathfrak{P}_n$. Using the second parameterization, Arnol'd interpreted the group $\mathcal{B}_n$ of n-braids as elements of the fundamental group $\pi_1(\mathfrak{P}_n, p)$ with base point p.

The conjugacy classes $\widehat{\mathcal{B}_n}$ of the braid group, equivalently, of the fundamental group $\pi_1(\mathfrak{P}_n, p)$, can be interpreted as free isotopy classes of loops in $\mathfrak{P}_n$. We define a collection of conformal invariants of the complex manifold $\mathfrak{P}_n$. Consider

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an element $\hat{b}$ of $\widehat{\mathcal{B}}_n$. We say that a continuous mapping f of an annulus $A=\{z\in\mathbb{C}:\ r<|z|<R\}$, $0\leqslant r<R\leqslant\infty$, into $\mathfrak{P}_n$ represents $\hat{b}$ if for some (and hence for any) circle $\{|z|=\rho\}\subset A$ the loop $f:\{|z|=\rho\}\to\mathfrak{P}_n$ represents $\hat{b}$. Ahlfors defined the conformal module of an annulus $A=\{z\in\mathbb{C}:\ r<|z|<R\}$, as $m(A)=\frac{1}{2\pi}\log(\frac{R}{r})$. Two annuli of finite conformal module are conformally equivalent iff they have equal conformal module. If a manifold Ω is conformally equivalent to an annulus A, its conformal module is defined to be $m(A)$. Associate with each conjugacy class of the fundamental group of $\mathfrak{P}_n$, or, equivalently, to each conjugacy class of n-braids, its conformal module, defined as follows.

Definition. Let $\hat{b}$ be a conjugacy class of n-braids, $n\geqslant 2$. The conformal module $M(\hat{b})$ of $\hat{b}$ is defined as $M(\hat{b})=\sup_{\mathcal{A}} m(A)$, where $\mathcal{A}$ denotes the set of all annuli that admit a holomorphic mapping into $\mathfrak{P}_n$ that represents $\hat{b}$.

For any complex manifold, the conformal module of conjugacy classes of its fundamental group can be defined. The collection of conformal modules of all conjugacy classes is a biholomorphic invariant of the manifold. This concept seems to be especially useful for locally symmetric spaces, for instance, for the quotient of the n-dimensional round complex ball by a subgroup of its automorphism group that acts freely and properly discontinuously. In this case, the universal covering is the ball and the fundamental group of the quotient manifold can be identified with the group of covering translations. For each covering translation, the problem is to consider the quotient of the ball by the action of the group generated by this single covering translation and to maximize the conformal module of annuli that admit holomorphic mappings into this quotient. The latter concept can be generalized to general mapping class groups. The generalization has relations to symplectic fibrations.

Runge's approximation theorem shows that the conformal module is positive for any conjugacy class of braids. The concept of the conformal module of conjugacy classes of braids appeared (without name) in the paper [7] which was motivated by the interest of the authors in Hilbert's Thirteen's Problem for algebraic functions.

The following objects related to $\mathfrak{P}_n$ have been considered in this connection. A continuous mapping of a (usually open and connected) Riemann surface X into the set of monic polynomials of degree n (maybe, with multiple zeros) is a quasipolynomial of degree n. It can be written as $P(x,\zeta)=a_0(x)+a_1(x)\zeta+\cdots+a_{n-1}(x)\zeta^{n-1}+\zeta^n$, $x\in X$, $\zeta\in\mathbb{C}$, for continuous functions a_j, $j=0,\dots,n-1$, on X. If the mapping is holomorphic, it is called an algebroid function. If the image of the map is contained in $\mathfrak{P}_n$, it is called separable. A separable quasipolynomial is called solvable if it can be globally written as a product of quasipolynomials of degree 1, and is called irreducible if it cannot be written as a product of two quasipolynomials of positive degree. Two separable quasipolynomials are isotopic if there is a continuous family of separable quasipolynomials joining them. An algebroid function on the complex line $\mathbb{C}$ whose coefficients are polynomials is called an algebraic function. A quasipolynomial P on X can be considered as a function on $X\times\mathbb{C}$. Its zero set $\mathfrak{S}_P=\{(x,\zeta)\in X\times\mathbb{C},\ P(x,\zeta)=0\}$ is a symplectic surface, called braided surface due to its relation to braids.

It will be useful to have the following result and terminology in mind. Let X be an open Riemann surface of finite genus with at most countably many ends. By [8], X is conformally equivalent to a domain Ω on a closed Riemann surface R such that the connected components of $R\setminus\Omega$ are all points or closed geometric discs. A geometric disc is a topological disc whose lift to the universal covering is a round disc (in the standard metric of the covering). If all connected components of $R\setminus\Omega$ are points, then X is called of first kind, otherwise it is called of second kind.

The conformal module of conjugacy classes of braids serves as obstruction for the existence of isotopies of separable quasipolynomials (respectively, of braided surfaces) to algebroid functions (respectively, to complex curves). Indeed, let X be an open Riemann surface. Suppose f is a separable quasipolynomial of degree n on X. Consider any domain $A\subset X$ which is conformally equivalent to an annulus. The restriction of f to A defines a mapping of the domain A into the space of polynomials $\mathfrak{P}_n$, hence it defines a conjugacy class of n-braids $\hat{b}_{f,A}$.

Lemma 1. *If f is algebroid, then $m(A)\leqslant M(\hat{b}_{f,A})$.*

Before giving examples of applications of the concept of the conformal module of conjugacy classes of braids, we compare this concept with a dynamical concept related to braids. Let $\mathbb{D}$ be the unit disc in the complex plane. Denote by E_n^0 the set consisting of the n points $0,\frac{1}{n},\dots,\frac{n-1}{n}$. Consider homeomorphisms of the n-punctured disc $\overline{\mathbb{D}}\setminus E_n^0$, which fix the boundary $\partial\mathbb{D}$ pointwise. Equivalently, these are homeomorphisms of the closed disc $\overline{\mathbb{D}}$ that fix the boundary pointwise and the set E_n^0 setwise. Equip this set of homeomorphisms with compact open topology. The connected components of this space form a group, called mapping class group of the n-punctured disc. This group is isomorphic to $\mathcal{B}_n$ [5]. Denote by $\mathcal{H}_b$ the connected component that corresponds to the braid b.

For a homeomorphism φ of a compact topological space, its topological entropy $h(\varphi)$ is an invariant that measures the complexity of its behavior under iterations. It is defined in terms of the action of the homeomorphism on open covers of the compact space. For a precise definition of topological entropy, we refer to the papers [1] or [6]. For a braid b, we define its entropy as $h(b)=\inf\{h(\varphi):\ \varphi\in\mathcal{H}_b\}$. The value is invariant under conjugation with self-homeomorphisms of the closed disc $\overline{\mathbb{D}}$, hence it does not depend on the position of the set of punctures and on the choice of the representative of the conjugacy class $\hat{b}$. We write $h(\hat{b})=h(b)$.

Entropy is a dynamical invariant. It has been considered in connection with Thurston's theory of surface homeomorphisms. Thurston himself used dynamical methods (Markov partitions) to show that many mapping class tori carry a complete hyperbolic metric of finite volume. Detailed proofs of Thurston's theorems are given in [6], where also the entropy of homeomorphisms of closed Riemann surfaces is studied. The study has been extended to Riemann surfaces with punctures. The common definition for braids is given in the irreducible case and uses mapping classes of the n-punctured complex plane rather than of the n-punctured disc. One can show that this definition is equivalent to the definition given above. Entropy has been studied intensively. E.g., the lowest non-vanishing entropy among irreducible braids on n strands, $n \geqslant 3$, has been estimated from below by $\frac{\log 2}{4} n^{-1}$ [9] and has been computed for small n. There is an algorithm for computing the entropy of irreducible braids (respectively, of irreducible mapping classes) [4]. Fluid mechanics related to stirring devises uses the entropy of the arising braids as a measure of complexity.

It turns out that the dynamical aspect and the conformal aspect are related. The following theorem holds.

Theorem 1. *For each $\hat{b} \in \widehat{\mathcal{B}_n}$ $(n \geqslant 2)$*

$$M(\hat{b}) = \frac{\pi}{2} \frac{1}{h(\hat{b})}.$$

The proof of the theorem uses Teichmüller's theory. The following known fact relies on Bers' paper [3], on Royden's theorem on equality of the Teichmüller metric and the Kobayashi metric on Teichmüller space [10], and on [6]. Let S be a closed Riemann surface or a finite open Riemann surface of the first kind. Suppose f is a self-homeomorphism of S, whose induced modular transformation f^* on the Teichmüller space $\mathcal{T}(S)$ of S is hyperbolic (in Thurston's terminology, the mapping class of f is pseudo-Anosov). Take the quotient of $\mathcal{T}(S)$ by the action of the group generated by f^*. Consider the largest conformal module of an annulus which admits a holomorphic mapping into the quotient such that the monodromy is f^*. Then the product of this conformal module with the entropy $h(f)$ equals $\frac{\pi}{2}$.

The space $\mathfrak{P}_n$ is not a Teichmüller space. However, it is related to the Teichmüller space of the $n+1$-punctured sphere as follows. The configuration space $C_n(\mathbb{C}) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n,\ z_i \neq z_j \text{ for } i \neq j\}$ is a holomorphic covering of $\mathfrak{P}_n$. Consider the quotient $C_n(\mathbb{C})/\mathcal{A}$ of the configuration space by the diagonal action of the group $\mathcal{A}$ of Möbius transformations that fix infinity. The Teichmüller space of the $n+1$-punctured sphere is the universal covering space of $C_n(\mathbb{C})/\mathcal{A}$. The covering projection is holomorphic. This relation allows one to associate with a holomorphic map f of an annulus into $\mathfrak{P}_n$ a modular transformation of the Teichmüller space. Analyzing the relation between this modular transformation and the conjugacy class of braids represented by f, and applying Royden's theorem and [3], one obtains the upper bound of the conformal module in the irreducible case. To obtain from a hyperbolic modular transformation a holomorphic mapping of an annulus into $\mathfrak{P}_n$, one needs to associate with a holomorphic mapping of a disc into the quotient $C_n(\mathbb{C})/\mathcal{A}$ a suitable holomorphic section to the configuration space. Its existence is based on the fact that an annulus A has trivial second cohomology $H^2(A, \mathbb{Z})$. The case of reducible braids is more subtle. Also, one has to consider Riemann surfaces of the second kind.

Corollary 1. *For each $\hat{b} \in \widehat{\mathcal{B}_n}$ $(n \geqslant 2)$ and each nonzero integer l*

$$M\big(\widehat{b^l}\big) = \frac{1}{|l|} M(\hat{b}).$$

Theorem 1 and [9] allow us to re-interpret and to reprove a result of [11] on conformal module, which we discuss for simplicity only for prime numbers n. Extending results of Lin and Gorin, Zjuzin [11] proved that a separable algebroid function $p(z)$, $z \in A$, of degree a prime number n on an annulus A is reducible if (a) the conformal module $m(A)$ is bigger than $n \cdot r_0$ for some absolute constant r_0 and (b) the index of the discriminant $D_n(p(z))$ is divisible by n. (The index is the degree of the mapping $z \to D_n(p(z)) \cdot |D_n(p(z))|^{-1}$ from a circle $\{|z| = \rho\} \subset A$ into the unit circle.) Indeed, if the quasipolynomial is irreducible, then the induced braid class $\hat{b}_{f,A}$ corresponds to the conjugacy class of n-cycles and, hence, $\hat{b}_{f,A}$ is represented by irreducible braids. Condition (a) with $r_0 = \frac{2\pi}{\log 2}$ implies by [9] and Theorem 1 that these braids must be periodic. But condition (b) excludes such periodic braids.

The following lemma gives a condition for solvability of algebroid functions on an annulus.

Lemma 2. *Let X be a closed Riemann surface of positive genus with a geometric disc removed. Suppose f is an irreducible separable algebroid function of degree 3 on X. Suppose X contains a domain A one of whose boundary components coincides with the boundary circle of X, such that A is conformally equivalent to an annulus of conformal module strictly larger than $\frac{\pi}{2}(\log(\frac{3+\sqrt{5}}{2}))^{-1}$. Then f is solvable over A.*

Note that $\log(\frac{3+\sqrt{5}}{2})$ is the smallest non-vanishing entropy among irreducible 3-braids. The estimate of the conformal module of A and the properties of the covering $\mathfrak{S}_f \to X$ allow only the solvable case or the case of periodic conjugacy classes $\hat{b}_{f,A}$ which correspond to conjugacy classes of 3-cycles. The latter is impossible for conjugacy classes of products of braid commutators.

Using Lemmas 1 and 2, we give the following application to algebroid functions on a torus with a hole.

Let X and Y be open Riemann surfaces and let f be a separable quasipolynomial on X. Let $w : X \to Y$ be a homeomorphism. The homeomorphism w can be interpreted as a new complex structure on X. Denote by f_w the quasipolynomial $f_w(y,z) = f(w^{-1}(y), \zeta)$, $y \in Y$, $\zeta \in \mathbb{C}$. We say that f is isotopic to an algebroid function for the complex structure w if f_w is isotopic to an algebroid function on Y.

Theorem 2. *Let X be a torus with a geometric disc removed. There exist eight conformal structures of the second kind on X with the following property. If f is an irreducible separable quasipolynomial of degree* 3 *on X, which is isotopic to an algebroid function for each of the eight conformal structures on X, then f is isotopic to an algebroid function for each complex structure on X, including complex structures of the first kind* (*determining punctured tori*)*.*

The conformal structures on the Riemann surface in Theorem 2 are chosen so that each contains a domain conformally equivalent to an annulus of conformal module estimated from below as in Lemma 2 and representing certain element of $\pi_1(X)$.

Let X be as in Theorem 2. Note that the fundamental group $\pi_1(X,x)$ of X with base point $x \in X$ is a free group on two generators. Consider a separable quasipolynomial of degree n and all its isotopies which fix the value at the base point x. For each element a of the fundamental group of $\pi_1(X,x)$, this defines a homotopy class of loops $\varphi(a)$ with a base point p in $\mathfrak{P}_n$, hence an n-braid. The mapping φ is a homomorphism from $\pi_1(X,x)$ to $\mathcal{B}_n$. There is a one-to-one correspondence between free isotopy classes of separable quasipolynomials (without fixing the value at a point) and conjugacy classes of homomorphisms from $\pi_1(X)$ to $\mathcal{B}_n$. The following theorem holds. It shows that the conditions of Theorem 2 are met very rarely.

Theorem 3. *Let X be a torus with a hole and f an irreducible separable quasipolynomial of degree* 3 *which is isotopic to an algebroid function for each complex structure on X. Then f corresponds to the conjugacy class of a homomorphism $\varphi : \pi_1(X) \to \Gamma$ where Γ is the subgroup of $\mathcal{B}_3$ generated by the periodic braid $\sigma_1\sigma_2$. Moreover, $\varphi(\pi_1(X))$ contains elements other than powers of the square $(\sigma_1\sigma_2)^3$ of the Garside element.*

Here σ_1 and σ_2 are the standard generators of $\mathcal{B}_3$. Associate with $\sigma_1\sigma_2$ the Möbius transformation $A(\zeta) = e^{\frac{-2\pi i}{3}} \cdot \zeta$, $\zeta \in \mathbb{P}^1$. The homomorphism φ in Theorem 3 defines a homomorphism h from $\pi_1(X)$ into the subgroup of the group of Möbius transformations that is generated by A. Hence, the fundamental group of any closed torus T acts freely and properly discontinuously on $\mathbb{C} \times \mathbb{P}^1$ by the holomorphic transformations $(z,\zeta) \to (\gamma(z), h(\gamma)(\zeta))$, $\gamma \in \pi_1(X)$. The quotient is the total space of a holomorphic line bundle over T. The bundle over a punctured torus is trivial and defines a separable algebroid function $\tilde{f}$ in the isotopy class of f. The set $\mathfrak{S}_{\tilde{f}}$ is a leaf of a holomorphic foliation of the total space of the bundle over the punctured torus.

There is also the concept of the conformal module of braids rather than of conjugacy classes of braids. This notion is based on the conformal module of rectangles which admit holomorphic mappings into $\mathfrak{P}_n$ with suitable boundary conditions on a pair of opposite sides. Recall that the conformal module of a rectangle with sides parallel to the coordinate axes is the ratio of the sidelengths of horizontal and vertical sides. The conformal module of braids is a finer invariant than entropy. If suitably defined it is more appropriate for application, in particular, to real algebraic geometry. In the case of three-braids there are two versions differing by the boundary conditions on horizontal sides of rectangles. (Both should be used.) For the first version, one requires that horizontal sides are mapped to polynomials with all zeros on a real line. In the second version, one requires that two of the zeros are at equal distance from the third. These are the cases appearing on the real axis for polynomials with real coefficients. The invariant can be studied by quasiconformal mappings and elliptic functions. The situation for braids on more than three strands is more subtle. We intend to come back to this concept in a later paper.

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