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Tauberian-type theorem for (e)-convergent sequences ☆

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ABSTRACT

We prove a Tauberian-type theorem for (e)-convergent sequences, which were introduced by the author in Karaev (2010) [4]. Our proof is based on the Berezin symbols technique of operator theory in the reproducing kernel Hilbert space.

1. Introduction

The notion of (e)-convergence for sequences was introduced by the author in [4]. Recall that (see [4]) if $\mathcal{H} = \mathcal{H}(\Omega)$ is a Hilbert space of complex-valued functions on some set Ω with reproducing kernel:

$$k_{\mathcal{H},\lambda}(z) := \sum_{n\geqslant 0} \overline{e_n(\lambda)} e_n(z),$$

where $e := (e_n(z))_{n\geqslant 0}$ is an orthonormal basis of $\mathcal{H}$, and $(a_n)_{n\geqslant 0}$ is any sequence of complex numbers, then we say that the sequence $(a_n)_{n\geqslant 0}$ is (e)-convergent to a if $\sum_{n=0}^{\infty} a_n |e_n(\lambda)|^2$ is convergent for all $\lambda \in \Omega$ and

$$\lim_{\lambda\to\xi} \frac{1}{\sum_{n=0}^{\infty} |e_n(\lambda)|^2} \sum_{n=0}^{\infty} a_n \left|e_n(\lambda)\right|^2 = a$$

for every $\xi \in \partial\Omega$.

It is easy to verify that for $\mathcal{H} = H^2(\mathbb{D})$ (Hardy space on the unit disc $\mathbb{D} = \{z \in \mathbb{C}\colon\ |z| < 1\}$ of the complex plane $\mathbb{C}$) and $\mathcal{H} = L_a^2(\mathbb{C})$ (Fock space) our definition of (e)-summability coincides with Abel and Borel summability, respectively.

We also recall that the so-called Berezin symbol of a bounded linear operator A on $\mathcal{H}$ is the function:

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_{\mathcal{H},\lambda}(z), \widehat{k}_{\mathcal{H},\lambda}(z)\right\rangle \quad (\lambda \in \Omega),$$

where $\widehat{k}_{\mathcal{H},\lambda}(z) := \frac{k_{\mathcal{H},\lambda}(z)}{\|k_{\mathcal{H},\lambda}\|_{\mathcal{H}}}$ is the normalized reproducing kernel of $\mathcal{H}$ (see, for instance, Berezin [1,2]). Clearly, $\|k_{\mathcal{H},\lambda}\|_{\mathcal{H}} = (\sum_{n=0}^{\infty} |e_n(\lambda)|^2)^{1/2}$.

For any bounded sequence $(a_n)_{n\geqslant 0}$ of complex numbers, let $D_{(a_k)}$ denote the diagonal operator on $\mathcal{H}$ defined by

$$D_{(a_k)} e_n(z) = a_n e_n(z), \quad n = 0, 1, 2, \ldots,$$

with respect to the orthonormal basis $e = (e_n(z))_{n\geqslant 0}$ of $\mathcal{H}$.

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Following Nordgren and Rosenthal [5], we say that a functional Hilbert space $\mathcal{H}(\Omega)$ is standard if the underlying set Ω is a subset of a topological space and if the boundary $\partial\Omega$ is non-empty and has the property that $(\widehat{k}_{\mathcal{H},\lambda})$ converges weakly to 0 whenever λ tends to a point on the boundary. For $\mathcal{H}(\Omega)$ a standard functional Hilbert space and A a compact operator on $\mathcal{H}(\Omega)$, $\widetilde{A}(\lambda)$ tends to 0 whenever λ tends to an a point on the boundary $\partial\Omega$ (since compact operators send weakly convergent sequences into strongly convergent ones). In this sense, the Berezin symbol of a compact operator on a standard functional Hilbert space vanishes on the boundary.

If $\{a_k\}_{k\geqslant 0}$ is a bounded sequence, then an easy calculus shows that (see [4]):

$$\widetilde{D}_{(a_k)}(\lambda) = \frac{1}{\sum_{n=0}^{\infty}|e_n(\lambda)|^2}\sum_{n=0}^{\infty} a_n \left|e_n(\lambda)\right|^2 \tag{1.1}$$

for all $\lambda\in\Omega$, which gives the following criteria for the (e)-summability method: the sequence $(a_k)_{k\geqslant 0}$ is (e)-convergent to a if and only if $\lim_{\lambda\to\xi}\widetilde{D}_{(a_k)}(\lambda) = a$ for every $\xi\in\partial\Omega$.

On the other hand, if $\mathcal{H}$ is a standard functional Hilbert space and $(a_k)_{k\geqslant 0}$ converges to a, then by considering that $\widetilde{D}_{(a_k)}(\lambda) = \widetilde{D}_{(a_k-a)} + a$ and $D_{(a_k-a)}$ is a compact operator on the standard functional Hilbert space $\mathcal{H}$, we obtain that (see formula (1.1)):

$$\begin{aligned} a &= \lim_{\lambda\to\xi}\bigl(\widetilde{D}_{(a_{k-a})}(\lambda) + a\bigr) = \lim_{\lambda\to\xi}\widetilde{D}_{(a_k)}(\lambda) \\ &= \lim_{\lambda\to\xi}\frac{1}{\sum_{n=0}^{\infty}|e_n(\lambda)|^2}\sum_{n=0}^{\infty} a_n\left|e_n(\lambda)\right|^2, \end{aligned}$$

which shows that $(a_k)_{k\geqslant 0}$ (e)-converges to a. This shows that the (e)-summability method associated with a standard functional Hilbert space is regular in the sense that it transforms convergent sequences into (e)-convergent sequences (see [4]).

The purpose of the present article is to solve a third important problem for (e)-convergence, namely, we will prove a Tauberian-type theorem for (e)-convergent sequences. (The good references for Tauberian-type theorems for valuable classical summability methods are the books by Hardy [3], by Powell and Shah [7], and also by Postnikov [6].)

2. A Tauberian-type theorem for (e)-convergent sequences

In this section, we use the Berezin symbols technique to prove a Tauberian-type theorem for (e)-convergent sequences. Namely, we will apply a result of Nordgren and Rosenthal [5, Corollary 2.8], which states that an operator on a standard functional Hilbert space is compact if and only if all the Berezin symbols of all unitarily equivalent operators vanish on the boundary.

Let $\mathcal{H} = \mathcal{H}(\Omega)$, $e = (e_n(z))_{n\geqslant 0}$ and $k_{\mathcal{H},\lambda}$ be the same as in the definition of (e)-convergent sequences. Also, let ℓ_1^2 denote the unit sphere of the sequences space l^2:

$$\ell_1^2 := \bigl\{(x_m)_{m\geqslant 0}\in\ell^2\colon\ \bigl\|(x_m)\bigr\|_{\ell^2} = 1\bigr\}.$$

Our result is the following.

Theorem. *Let $(a_n)_{n\geqslant 0}$ be a bounded sequence of complex numbers such that $(a_n)_{n\geqslant 0}$ (e)-converges to a. Denoting by $k_{\mathcal{H},\lambda}$ the reproducing kernel of the standard functional Hilbert space $\mathcal{H}$ at λ, we assume that*

$$\sum_{m=0}^{+\infty}(a_m - a)\left|\sum_{n=0}^{+\infty}\bar{x}_{mn}e_n(\lambda)\right|^2 = o\quad\bigl(\|k_{\mathcal{H},\lambda}\|^2\bigr) \tag{2.1}$$

for every double sequence $(x_{mn})_{m,n=0}^{+\infty}$ with $\|(x_{mn})_m\| = 1$ $(\forall n\geqslant 0)$ and $\|(x_{mn})_n\| = 1$ $(\forall m\geqslant 0)$, whenever λ tends to a point in the boundary of Ω. Then $(a_n)_n$ converges to a in the usual sense.

Proof. First note that if U is a unitary operator on $\mathcal{H}$ and $b_{mn} := \langle Ue_n(z), e_m(z)\rangle$ $(m,n = 0,1,2,\ldots)$ are its matrix elements, then it is classical (and trivial) that both sequences $(b_{mn})_m$ and $(b_{mn})_n$ are in the unit sphere ℓ_1^2 of ℓ^2. Then we have

$$\begin{aligned} \widetilde{U^{-1}D_{(a_n-a)}U}(\lambda) &= \bigl\langle U^{-1}D_{(a_n-a)}U\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\bigr\rangle \\ &= \frac{1}{\|k_{\mathcal{H},\lambda}\|^2}\left\langle U^{-1}D_{(a_n-a)}U\sum_{n\geqslant 0}\overline{e_n(\lambda)}e_n(z), \sum_{n\geqslant 0}\overline{e_n(\lambda)}e_n(z)\right\rangle \\ &= \frac{1}{\|k_{\mathcal{H},\lambda}\|^2}\left\langle D_{(a_n-a)}\sum_{n\geqslant 0}\overline{e_n(\lambda)}Ue_n(z), \sum_{n\geqslant 0}\overline{e_n(\lambda)}Ue_n(z)\right\rangle \end{aligned}$$

$$= \frac{1}{\|k_{\mathcal{H},\lambda}\|^2}\left\langle D_{(a_n-a)} \sum_{n\geqslant 0} \overline{e_n(\lambda)} \sum_{m\geqslant 0} b_{mn} e_m(z), \sum_{n\geqslant 0} \overline{e_n(\lambda)} \sum_{m\geqslant 0} b_{mn} e_m(z)\right\rangle$$
$$= \frac{1}{\|k_{\mathcal{H},\lambda}\|^2}\left\langle \sum_{n\geqslant 0}\sum_{m\geqslant 0} \overline{e_n(\lambda)}(a_m-a) b_{mn} e_m(z), \sum_{n\geqslant 0}\sum_{m\geqslant 0} \overline{e_n(\lambda)} b_{mn} e_m(z)\right\rangle$$
$$= \frac{1}{\|k_{\mathcal{H},\lambda}\|^2}\left\langle \sum_{m\geqslant 0}(a_m-a)\left(\sum_{n\geqslant 0} \overline{e_n(\lambda)} b_{mn}\right) e_m(z), \sum_{m\geqslant 0}\left(\sum_{n\geqslant 0} \overline{e_n(\lambda)} b_{mn}\right) e_m(z)\right\rangle$$
$$= \frac{1}{\|k_{\mathcal{H},\lambda}\|^2}\sum_{m=0}^{\infty}(a_m-a)\left|\sum_{n=0}^{\infty}\bar{b}_{mn} e_n(\lambda)\right|^2$$

for every unitary operator U and $\lambda \in \Omega$.

Hence, by using condition (2.1) of the theorem, we assert that

$$U^{-1}\widetilde{D_{(a_n-a)}U}(\lambda) \to 0$$

as λ tends to a point on the boundary $\partial\Omega$, as desired. Then, by the above-mentioned result of Nordgren and Rosenthal [5, Corollary 2.8], we deduce that $D_{(a_n-a)}$ is a compact operator on $\mathcal{H}$, and consequently $a_n - a \to 0$ $(n \to \infty)$, that is a_n converges to a, which proves the theorem. □

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