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Analytic Geometry

Generalized L^2 extension theorem and a conjecture of Ohsawa ☆

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ABSTRACT

In this paper, we determine the optimal constant in the estimate of Ohsawa's generalized L^2 extension theorem. The result holds for holomorphic vector bundles on a class of complex manifolds including both Stein manifolds and complex projective algebraic manifolds. As an application, we obtain a solution to a related conjecture of Ohsawa.

RÉSUMÉ

Dans cet article, nous déterminons la constante optimale intervenant dans l'estimée du théorème d'extension L^2 généralisé de Ohsawa. Le résultat vaut pour les fibrés vectoriels holomorphes sur une classe de variétés complexes incluant à la fois les variétés de Stein et les variétés algébriques projectives complexes. Comme application, nous obtenons la solution d'une conjecture correspondante d'Ohsawa.

1. Background and notations

L^2 method for solving $\bar\partial$ equation is important and developed quickly. A starting point is Hörmander's L^2 existence theorem (see [5]). In [20], Ohsawa proved a general L^2 extension theorem in order to cover all earlier results of him in [21,17–19] in this direction. Before stating Ohsawa's main theorem in [20], we would like to recall the symbols and notations which he used. One considers the following data (M, S): a complex n-dimensional manifold M, a closed complex submanifold S of M, such that there exists a closed subset $X \subset M$ of Lebesgue measure 0 satisfying:

a) X is locally negligible with respect to L^2 holomorphic functions, i.e., for any local coordinate neighborhood $U \subset M$ and for any L^2 holomorphic function f on $U \setminus X$, there exists an L^2 holomorphic function $\tilde f$ on U such that $\tilde f|_{U\setminus X} = f$;
b) $M \setminus X$ is a Stein manifold which intersects with every component of S.

For the sake of convenience, we say that (M, S) *satisfy conditions* $a), b)$. Such a kind of manifolds includes:

1) Stein manifolds (including open Riemann surfaces);
2) Complex projective algebraic manifolds (including compact Riemann surfaces);
3) Projective families (see [24,9]).

Let dV_M be a continuous volume form on M. One considers a class of upper semi-continuous function Ψ from M to the interval $[-\infty, 0)$ such that

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1) $\Psi^{-1}(-\infty) \supset S$, and $\Psi^{-1}(-\infty)$ is a closed subset of M;
2) If S is l-dimensional around a point x, there exists a local coordinate $(z_1, \dots, z_n)$ on a neighborhood U of x such that $z_{l+1} = \dots = z_n = 0$ on $S \cap U$ and

$$\sup_{U \setminus S} \left| \Psi(z) - (n-l) \log \sum_{l+1}^{n} |z_j|^2 \right| < \infty.$$

The set of such polar function Ψ will be denoted by $\#(S)$.

For each $\Psi \in \#(S)$, one can associate a positive measure $\mathrm{d}V_M[\Psi]$ on S as the minimum element of the partial ordered set of positive measures $\mathrm{d}\mu$ satisfying:

$$\int_{S_l} f \, \mathrm{d}\mu \geqslant \limsup_{t \to \infty} \frac{2(n-l)}{\sigma_{2n-2l-1}} \int_M f e^{-\Psi} \mathbb{1}_{\{-1-t<\Psi<-t\}} \, \mathrm{d}V_M$$

for any non-negative continuous function f with compact support on M. Throughout the paper S_l denotes the l-dimensional component of S, σ_m denotes the volume of the unit sphere in $\mathbb{R}^{m+1}$.

Let u be a continuous section of $K_M \otimes E$, where E is a holomorphic vector bundle equipped with a continuous metric h on M. We define

$$|u|_h^2\big|_V := \frac{c_n h(e,e) v \wedge \bar{v}}{\mathrm{d}V_M},$$

where $c_n = (-1)^{\frac{n(n-1)}{2}} i^n$, and $u|_V = v \otimes e$ for an open set $V \subset M \setminus X$, v is a continuous section of $K_M|_V$ and e is a continuous section of $E|_V$ (especially, we define

$$|v|^2\big|_V := \frac{c_n v \wedge \bar{v}}{\mathrm{d}V_M},$$

when v is a continuous section of K_M). It is clear that $|u|_h^2$ is independent of the choice of V, while $|u|_h^2 \, \mathrm{d}V_M$ is independent of the choice of $\mathrm{d}V_M$. Then the space of L^2 integrable holomorphic section of K_M is denoted by $A^2(M, K_M, \mathrm{d}V_M^{-1}, \mathrm{d}V_M)$. Respectively, the space of holomorphic section of $K_M|_S$ which is L^2 integrable with respect to the measure $\mathrm{d}V_M[\Psi]$ is denoted by $A^2(S, K_M|_S, \mathrm{d}V_M^{-1}, \mathrm{d}V_M[\Psi])$.

Let $\Delta(S)$ be the subset of plurisubharmonic functions in $\#(S)$, φ be a locally integrable function on M. Let $\Delta_{\varphi,\delta}(S)$ be the subset of functions Ψ in $\#(S)$, such that $\Psi + \varphi$ and $(1+\delta)\Psi + \varphi$ are both plurisubharmonic functions on M.

Let $\Delta_{h,\delta}(S)$ be the subset of functions Ψ in $\#(S)$ which satisfies $\Theta_{he^{-\Psi}} \geqslant 0$ and $\Theta_{he^{-(1+\delta)\Psi}} \geqslant 0$ on $M \setminus S$ in the sense of Nakano.

Theorem 1.1. *(See [20].) Let (M, S) satisfy conditions a), b), h be a smooth metric on a holomorphic vector bundle E on M with rank r. Then, for any function Ψ on M such that $\Psi \in \Delta_{h,\delta}(S) \cap C^\infty(M \setminus S)$, there exists a uniform constant $\mathbf{C} = \max_{1 \leqslant k \leqslant n} \frac{2^8 \pi}{\frac{\pi^k}{k!}}$ such that, for any holomorphic section f of $K_M \otimes E|_S$ on S satisfying*

$$\sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 \, \mathrm{d}V_M[\Psi] < \infty,$$

there exists a holomorphic section F of $K_M \otimes E$ on M satisfying $F = f$ on S and

$$\int_M |F|_h^2 \, \mathrm{d}V_M \leqslant \mathbf{C}\big(1 + \delta^{-\frac{3}{2}}\big) \sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 \, \mathrm{d}V_M[\Psi].$$

Especially, if $\Psi \in \Delta(S) \cap \Delta_{h,\delta}(S) \cap C^\infty(M \setminus S)$, there exists a holomorphic section F of $K_M \otimes E$ on M satisfying $F = f$ on S and

$$\int_M |F|_h^2 \, \mathrm{d}V_M \leqslant \mathbf{C} \sum_{k=1}^{n} \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 \, \mathrm{d}V_M[\Psi].$$

A natural interesting problem is to find the optimal constant $\mathbf{C}$ and the optimal power of δ in Theorem 1.1, which can be used to prove a conjecture of Ohsawa as an application (see Section 4). In the setting of Ohsawa–Takegoshi [21], the problem for analytic hypersurfaces was widely discussed for various cases in [21,15,18,23,1,7,24,4,16,10,2,3]. In the setting of Ohsawa [18], continuing our work in [14,26], we solved the problem in [12], as a corollary, we obtained a solution of Suita's conjecture [12]. Blocki solved Suita's conjecture for bounded planar domains (see [2,3]).

In the present note, we obtain the optimal constant version of Theorem 1.1. We state our main results in Sections 2 and 3 respectively for two cases: one is on holomorphic vector bundles and polar function Ψ smooth outside S, and another is on holomorphic line bundles and upper semi-continuous polar function Ψ. Consequently, we solve a conjecture of Ohsawa. Details will appear in [13]. In the proof of the main results, a theorem in [11] on smoothing of plurisubharmonic functions is used.

2. Optimal constant problem in the generalized L^2 extension theorem on vector bundles

Considering holomorphic vector bundles and polar function Ψ smooth outside S, we establish the following optimal constant version of Theorem 1.1:

Theorem 2.1. *Theorem 1.1 holds true with the control constant* $\mathbf{C}(1+\delta^{-1})$ *where* $\mathbf{C}=1$ *in the* L^2 *estimate. Especially, if* Ψ *is furthermore plurisubharmonic, one gets the control constant* $\mathbf{C}=1$ *in the* L^2 *estimate.*

Remark 2.2. $\mathbf{C}$ and the power of δ are both optimal. For example, we may take the unit disc with trivial holomorphic line bundle while $S=\{0\}$.

3. Optimal constant problem in the generalized L^2 extension theorem on line bundles

Considering holomorphic line bundles with singular metric and upper semi-continuous polar functions Ψ, we establish the following optimal constant version of Theorem 1.1, which will be used to prove a conjecture of Ohsawa in Section 4.

Theorem 3.1. *Let* (M,S) *satisfy conditions* $a), b)$, *L be a holomorphic line bundle on M with a continuous metric h (resp. a singular metric h satisfying $\Theta_h \geqslant \omega$, where ω is a smooth real $(1,1)$ form on M). Then, for negative function Ψ on M satisfying $\Theta_{he^{-\Psi}} \geqslant 0$ and $\Theta_{he^{-(1+\delta)\Psi}} \geqslant 0$ (resp. $\sqrt{-1}\partial\bar\partial\Psi+\omega>0$ and $(1+\delta)\sqrt{-1}\partial\bar\partial\Psi+\omega>0$) in the sense of current on M, there exists a uniform constant $\mathbf{C}=1$, such that, for any holomorphic section f of $K_M\otimes L|_S$ on S satisfying $\sum_{k=1}^n \frac{\pi^k}{k!}\int_{S_{n-k}}|f|_h^2\,\mathrm{d}V_M[\Psi]<\infty$, there exists a holomorphic section F of $K_M\otimes L$ on M satisfying $F=f$ on S and*

$$\int_M |F|_h^2\,\mathrm{d}V_M \leqslant \mathbf{C}\big(1+\delta^{-1}\big)\sum_{k=1}^n \frac{\pi^k}{k!}\int_{S_{n-k}} |f|_h^2\,\mathrm{d}V_M[\Psi].$$

Especially, if L is a holomorphic line bundle on M with a singular metric h satisfying $\Theta_h \geqslant 0$ in the sense of current, then, for a negative plurisubharmonic function Ψ on M satisfying $\Psi\in\Delta(S)$, there exists a uniform constant $\mathbf{C}=1$, such that there exists a holomorphic section F of $K_M\otimes L$ on M satisfying $F=f$ on S and

$$\int_M |F|_h^2\,\mathrm{d}V_M \leqslant \mathbf{C}\sum_{k=1}^n \frac{\pi^k}{k!}\int_{S_{n-k}} |f|_h^2\,\mathrm{d}V_M[\Psi].$$

4. A conjecture of Ohsawa

If $\Delta(S)$ is non-empty, we set $G(z,S):=\sup\{u(z)\colon\ u\in\Delta(S)\}^*$, which is the upper envelope of $\sup\{u(z)\colon\ u\in\Delta(S)\}$. It is clear that $G(z,S)$ is a plurisubharmonic function on M (see Choquet's Lemma (Lemma 4.23 in [8])). By Proposition 9 in [20], we have $G(z,S)\in\Delta(S)$. If $\Delta(S)$ is empty, $G(z,S):=-\infty$. When $S=\{z\}$ for some $z\in M$, $G(z,S)$ is called the pluricomplex Green function (see [6]).

Let (M,S) satisfy conditions $a), b)$, $G(\cdot,S)$ be the generalized pluricomplex Green function, which is nontrivial. Let $\mathrm{d}V_M$ be a continuous volume form on M and let $\{\sigma_j\}_{j=1}^\infty$ (resp. $\{\tau_j\}_{j=1}^\infty$) be a complete orthogonal system of $A^2(M,K_M,\mathrm{d}V_M^{-1},\mathrm{d}V_M)$ (resp. $A^2(S,K_M|_S,\mathrm{d}V_M^{-1},\mathrm{d}V_M[G(\cdot,S)])$) and put $\kappa_M=\sum_{j=1}^\infty \sigma_j\otimes\bar\sigma_j\in C^\omega(M,K_M\otimes\bar K_M)$ (resp. $\kappa_{M/S}=\sum_{j=1}^\infty \tau_j\otimes\bar\tau_j\in C^\omega(S,K_M\otimes\bar K_M)$).

Estimating constant $\mathbf{C}$ in Theorem 1.1 is motivated by the following conjecture of Ohsawa (see [20]) on (M,S) satisfying conditions $a), b)$, which admits nontrivial generalized pluricomplex Green functions with poles on S:

A conjecture of Ohsawa. $(\pi^k/k!)\kappa_M(x)\geqslant\kappa_{M/S}(x)$ *for any* $x\in S_{n-k}$.

The relationship between the conjecture of Ohsawa and the extension theorem was observed and explored by Ohsawa [20], who proved the estimate with $\mathbf{C}=\frac{2^8\pi}{\pi^k/k!}$. The conjecture of Ohsawa can be seen as an extension of Suita's conjecture (see [25]) for high dimensional manifolds and high codimensional submanifolds (see Section 3 of [20]).

Using Theorem 3.1, we get:

Corollary 4.1 *(Solution of the conjecture of Ohsawa). Let (M,S) satisfy conditions a), b), where M admits a nontrivial generalized pluricomplex Green function with poles on S. Then we have $\mathbf{C}(\pi^k/k!)\kappa_M(x) \geqslant \kappa_{M/S}(x)$ for any $x \in S_{n-k}$, where the constant satisfies $\mathbf{C} = 1$.*

Remark 4.2. It is known that $\kappa_{M/S}(x) = c_M^2(z)|\mathrm{d}z|^2$ when M is an open Riemann surface which admit a Green function and $S = \{z\}$ (see Section 3 of [20]), where $c_M(z)$ be the logarithmic capacity of M with respect to z locally defined by $c_M(z) = \exp\lim_{\xi\to z}(G_M(\xi,z) - \log|\xi - z|)$, where G_M is the negative Green function on M. The conjecture of Ohsawa includes Suita's conjecture as a one-dimensional case. Suita's conjecture was posed in [25] originally for open Riemann surfaces admitting Green functions, which is a conjectural answer to an open question posed by Sario and Oikawa (see pages 179 and 342 in [22]) about the relation between the Bergman kernel and logarithmic capacity on open Riemann surfaces.

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