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Hardy–Littlewood's inequalities in the case of a capacity

Inégalités de Hardy–Littlewood dans le cas d'une capacité

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ABSTRACT

Hardy–Littlewood's inequalities, well known in the case of a probability measure, are extended to the case of a monotone (but not necessarily additive) set function, called a capacity. The upper inequality is established in the case of a capacity assumed to be continuous and submodular, the lower — under assumptions of continuity and supermodularity.

RÉSUMÉ

Sous des hypothèses appropriées, nous généralisons les inégalités de Hardy–Littlewood, bien connues dans le cas où l'espace mesurable sous-jacent est muni d'une probabilité, au cas d'une fonction d'ensembles monotone, appelée capacité. Le résultat fait usage de la théorie de l'intégration au sens de Choquet.

1. Some definitions and basic properties

The definitions and results recalled in this section can be found in the book by D. Denneberg [1], and/or in that by H. Föllmer and A. Schied (cf. [2, Section 4.7]).

Let $(\Omega, \mathcal{F})$ be a measurable space.

Definition 1.1. A set function $\mu : \mathcal{F} \to [0,1]$ is called a *capacity* if it satisfies $\mu(\varnothing) = 0$ (groundedness), $\mu(\Omega) = 1$ (normalization) and the following monotonicity property: $A, B \in \mathcal{F}, A \subset B \Rightarrow \mu(A) \leqslant \mu(B)$.

A capacity μ is called *submodular* (or concave, or 2-alternating) if

$$\mu(A \cup B) + \mu(A \cap B) \leqslant \mu(A) + \mu(B), \quad \text{for all } A, B \in \mathcal{F}.$$

A capacity μ is called *supermodular* (or convex) if it satisfies the previous property where the inequality is reversed.
A capacity μ is called *continuous from below* if

$$(A_n) \subset \mathcal{F} \quad \text{such that} \quad A_n \subset A_{n+1},\ \forall n \in \mathbb{N} \quad \Rightarrow \quad \lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right).$$

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Definition 1.2. Two real-valued measurable functions X and Y on $(\Omega, \mathcal{F})$ are called comonotonic if

$$\big(X(\omega) - X(\omega')\big)\big(Y(\omega) - Y(\omega')\big) \geqslant 0, \quad \forall(\omega, \omega') \in \Omega \times \Omega.$$

For a non-negative measurable function X on $(\Omega, \mathcal{F})$, the *Choquet integral* of X with respect to a capacity μ is defined as follows:

$$\mathbb{E}_\mu(X) := \int_0^{+\infty} \mu(X > x)\,\mathrm{d}x.$$

Let X and Y be two non-negative measurable functions on $(\Omega, \mathcal{F})$. The Choquet integral with respect to a capacity μ has the following properties:

- (*positive homogeneity*) $\mathbb{E}_\mu(\lambda X) = \lambda \mathbb{E}_\mu(X), \forall \lambda \in \mathbb{R}_+$
- (*monotonicity*) $X \leqslant Y \Rightarrow \mathbb{E}_\mu(X) \leqslant \mathbb{E}_\mu(Y)$
- (*comonotonic additivity*) If X and Y are comonotonic, then $\mathbb{E}_\mu(X + Y) = \mathbb{E}_\mu(X) + \mathbb{E}_\mu(Y)$.

Moreover, if the capacity μ is assumed to be *submodular*, then the following *subadditivity* property holds:

- (*subadditivity*) $\mathbb{E}_\mu(X + Y) \leqslant \mathbb{E}_\mu(X) + \mathbb{E}_\mu(Y)$.

The reader is referred to [1] for the following result.

Theorem 1.3 *(Monotone convergence). Let μ be a capacity on $(\Omega, \mathcal{F})$ which is continuous from below. For a non-decreasing sequence (X_n) of non-negative measurable functions, we have*:

$$\lim_{n\to\infty} \mathbb{E}_\mu(X_n) = \mathbb{E}_\mu(\lim_{n\to\infty} X_n).$$

We recall the notions of (non-decreasing) distribution function and of a quantile function with respect to a capacity μ (cf. [2]).

Definition 1.4. Let X be a measurable function on $(\Omega, \mathcal{F})$. We define the distribution function G_X of X with respect to μ by $G_X(x) := 1 - \mu(X > x)$, for all $x \in \bar{\mathbb{R}}$.

Any *generalized inverse* function $r_X : (0, 1) \to \bar{\mathbb{R}}$ of the *non-decreasing* function G_X is called a quantile function of X with respect to μ.

The following properties of quantile functions with respect to a capacity are well known (cf. [1]):

(Q1) If $\lambda \geqslant 0$, then $r_{\lambda X}(t) = \lambda r_X(t)$, for almost every $t \in (0, 1)$.
(Q2) If X, Y is a pair of (real-valued) comonotonic functions, then $r_{X+Y}(t) = r_X(t) + r_Y(t)$, for almost every t.

2. Hardy–Littlewood's inequalities in the case of a capacity

We state the main result of the present note. For the corresponding result in the particular case where μ is a probability measure, the reader is referred to Theorem A.24 in [2] and the references therein.

Theorem 2.1 *(Hardy–Littlewood's inequalities). Let μ be a capacity on $(\Omega, \mathcal{F})$. Let X and Y be two non-negative measurable functions with quantile functions (with respect to the capacity μ) denoted by r_X and r_Y.*

(i) *If μ is submodular and continuous from below, then $\mathbb{E}_\mu(XY) \leqslant \int_0^1 r_X(t) r_Y(t)\,\mathrm{d}t$.*
(ii) *If μ is supermodular and continuous from below, then $\mathbb{E}_\mu(XY) \geqslant \int_0^1 r_X(1-t) r_Y(t)\,\mathrm{d}t$.*

The proof is based on the following lemma.

Lemma 2.2. *Let μ be a capacity on $(\Omega, \mathcal{F})$ which is continuous from below. Let (X_n) be a non-decreasing sequence of non-negative measurable functions and let X denote the limit function.*

(i) *The sequence of distribution functions (with respect to μ) of X_n is non-increasing and converges to the distribution function (with respect to μ) of X, i.e. $G_{X_n}(x) \downarrow G_X(x)$, for all $x \in \bar{\mathbb{R}}_+$.*

(ii) *The following convergence holds as well*: $r_{X_n}(t) \uparrow r_X(t)$ *for almost every* t, *where* r_{X_n} *and* r_X *stand for* (*versions of*) *the quantile functions* (*with respect to* μ) *of* X_n *and* X, *respectively.*

Proof. The proof of the first statement is contained in the proof of Theorem 8.1 in [1].

To prove the second statement, we will use the lower quantile function $r^l_{X_n}$ of X_n defined by:

$$r^l_{X_n}(t) := \sup\{x \in \mathbb{R}\colon\ G_{X_n}(x) < t\}, \quad \text{for } t \in (0,1).$$

As the sequence (X_n) is non-negative, non-decreasing, the sequence $(r^l_{X_n})$ is non-negative, non-decreasing; we denote by r the limit function of the latter, i.e. $r(t) := \lim_n r^l_{X_n}(t) = \sup_n r^l_{X_n}(t), \forall t \in (0,1)$. We will show that for all $t \in (0,1), r(t) = r^l_X(t)$, where $r^l_X(t) := \sup\{x \in \mathbb{R} : G_X(x) < t\}$ is the lower quantile function of X (with respect to μ). The conclusion of the lemma will follow as $r^l_X = r_X$ almost everywhere and $r^l_{X_n} = r_{X_n}$ almost everywhere.

Now, $G_{X_n} \geqslant G_X$ for all n, which implies that $r^l_{X_n}(t) \leqslant r^l_X(t), \forall t \in (0,1), \forall n$. By passing to the limit, we obtain $r(t) \leqslant r^l_X(t), \forall t \in (0,1)$.

We turn to the proof of the converse inequality, namely $r(t) \geqslant r^l_X(t), \forall t \in (0,1)$. Fix $t \in (0,1)$ and let $x \in \mathbb{R}$ be such that $G_X(x) < t$. By the first part of the lemma, we know that $G_{X_n}(x) \downarrow G_X(x)$. Hence, there exists $n_0 = n_0(t,x)$ such that for all $n \geqslant n_0$, $G_{X_n}(x) < t$. Therefore, for all $n \geqslant n_0$, $x \in \{y \in \mathbb{R}\colon\ G_{X_n}(y) < t\}$ which implies that $r^l_{X_n}(t) := \sup\{y \in \mathbb{R}\colon\ G_{X_n}(y) < t\} \geqslant x, \forall n \geqslant n_0$. By passing to the limit, we obtain that $r(t) \geqslant x$, which gives the desired inequality and concludes the proof. □

Proof of Theorem 2.1. We will prove the first part of the theorem which concerns the upper bound. The lower bound can be proved by means of similar arguments.

Step 1. The inequality is satisfied by X and Y of the form $X = \mathbb{I}_A$, $Y = \mathbb{I}_B$, where $A, B \in \mathcal{F}$ (even without the assumption of continuity from below and submodularity of μ). Indeed,

$$\mathbb{E}_\mu(\mathbb{I}_A \mathbb{I}_B) = \mu(A \cap B) \leqslant \mu(A) \wedge \mu(B) = \int_0^1 r_{\mathbb{I}_A}(t) r_{\mathbb{I}_B}(t)\,\mathrm{d}t, \tag{1}$$

where we have used that $r_{\mathbb{I}_A} = \mathbb{I}_{(1-\mu(A),1]}$ a.e. in order to obtain the last equality in (1).

Step 2. We prove the desired inequality for non-negative step functions. Let X and Y be two non-negative step functions. The function X has the following representation $X = \sum_{i=1}^n x_i \mathbb{I}_{A_i}$, with $x_i \geqslant 0$ and $A_i \in \mathcal{F}$. Without loss of generality, we can suppose that the numbers x_i are ranged in a descending order (i.e. $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0$) and that the sets A_i are disjoint. Thus, the function X can be rewritten in the following manner: $X = \sum_{i=1}^n \tilde{x}_i \mathbb{I}_{\tilde{A}_i}$, where $\tilde{x}_i := x_i - x_{i+1} \geqslant 0$, $x_{n+1} := 0$ and $\tilde{A}_i := \bigcup_{k=1}^i A_k$. We note that the functions $\tilde{x}_i \mathbb{I}_{\tilde{A}_i}$ and $\tilde{x}_j \mathbb{I}_{\tilde{A}_j}$ are comonotonic. In the same manner, the function Y has the following representation: $Y = \sum_{j=1}^m \tilde{y}_j \mathbb{I}_{\tilde{B}_j}$, where $\tilde{y}_j \geqslant 0$ and $\tilde{B}_j \subset \tilde{B}_{j+1}$.

Thanks to the subadditivity of the Choquet integral with respect to a submodular capacity and to the positive homogeneity of the Choquet integral, we have:

$$\mathbb{E}_\mu(XY) \leqslant \sum_{i=1}^n \sum_{j=1}^m \tilde{x}_i \tilde{y}_j \mu(\tilde{A}_i \cap \tilde{B}_j). \tag{2}$$

On the other hand, we see that $r_X = \sum_{i=1}^n r_{X_i}$ a.e. where we have set $X_i := \tilde{x}_i \mathbb{I}_{\tilde{A}_i}$ and where r_{X_i} designates a quantile function of X_i. Indeed, as mentioned above, the functions in the sum $\sum_{i=1}^n \tilde{x}_i \mathbb{I}_{\tilde{A}_i}$ are pairwise comonotonic; therefore, the functions $\sum_{i=1}^{k-1} \tilde{x}_i \mathbb{I}_{\tilde{A}_i}$ and $\tilde{x}_k \mathbb{I}_{\tilde{A}_k}$ are comonotonic; property (Q2) and a reasoning by induction allow us to conclude. By the same arguments, $r_Y = \sum_{j=1}^m r_{Y_j}$ a.e. where $Y_j := \tilde{y}_j \mathbb{I}_{\tilde{B}_j}$ and r_{Y_j} designates a quantile function of Y_j. So,

$$\int_0^1 r_X(t) r_Y(t)\,\mathrm{d}t = \sum_{i=1}^n \sum_{j=1}^m \tilde{x}_i \tilde{y}_j \int_0^1 r_{\mathbb{I}_{\tilde{A}_i}}(t) r_{\mathbb{I}_{\tilde{B}_j}}(t)\,\mathrm{d}t, \tag{3}$$

where the non-negativity of $\tilde{x}_i$ and $\tilde{y}_j$ and property (Q1) have been used.

From the first step of the proof about indicator functions, we know that $\mu(\tilde{A}_i \cap \tilde{B}_j) \leqslant \int_0^1 r_{\mathbb{I}_{\tilde{A}_i}}(t) r_{\mathbb{I}_{\tilde{B}_j}}(t)\,\mathrm{d}t$ (cf. Eq. (1)). The second step is proved, by combining this observation with Eqs. (2) and (3).

Step 3. To prove the inequality in the general case, let X and Y be two measurable non-negative functions. Let (X_n) be a sequence of non-negative step functions such that $X_n \uparrow X$, and let (Y_n) be a sequence of non-negative step functions such that $Y_n \uparrow Y$. From the second step of the proof, we know that $\mathbb{E}_\mu(X_n Y_n) \leqslant \int_0^1 r_{X_n}(t) r_{Y_n}(t)\,\mathrm{d}t$, for all n. By applying the monotone convergence theorem (Theorem 1.3) to the non-negative, non-decreasing sequence $(X_n Y_n)$, we obtain

$\lim_{n\to\infty} \mathbb{E}_\mu(X_n Y_n) = \mathbb{E}_\mu(XY)$. On the other hand, by using Lemma 2.2, we obtain $r_{X_n}(t) \uparrow r_X(t)$ for almost every t and $r_{Y_n}(t) \uparrow r_Y(t)$ for almost every t; these considerations, along with the non-negativity of $r_{X_n}(\cdot)$ and $r_{Y_n}(\cdot)$ for all n, lead to $r_{X_n}(t) r_{Y_n}(t) \uparrow r_X(t) r_Y(t)$ for almost every t. The monotone convergence theorem for Lebesgue integrals, applied to the sequence $(r_{X_n}(\cdot) r_{Y_n}(\cdot))$, gives $\lim_{n\to\infty} \int_0^1 r_{X_n}(t) r_{Y_n}(t)\,\mathrm{d}t = \int_0^1 r_X(t) r_Y(t)\,\mathrm{d}t$, which concludes the proof. □

We note that, as in the particular case where μ is a probability measure, the upper bound in Theorem 2.1 is attained by a pair of non-negative comonotonic measurable functions. We remark, as well, that a result analogous to Theorem 2.1 can be established in the case where $\mu(\Omega)$ is finite, but not necessarily normalized to 1.

In the case where the measurable functions can take negative values, Theorem 2.1 does not necessarily hold true, as can be seen from the following counter-example. For the definition of the (asymmetric) Choquet integral in this case, the reader is referred to Chapter 5 in [1], and to [2]. Let $(\Omega, \mathcal{F}, \mu)$ be given, where μ is a non-additive submodular (resp. *supermodular*) capacity. Then, there exists $A \in \mathcal{F}$ such that $\mu(A) >$ (resp. $<$) $1 - \mu(A^c)$. We set $X := \mathbb{I}_A$ and $Y := b$, where $b < 0$. An explicit computation gives $\mathbb{E}_\mu(XY) = b(1 - \mu(A^c))$ and $\int_0^1 r_X(t) r_Y(t)\,\mathrm{d}t = \int_0^1 r_X(t) r_Y(1-t)\,\mathrm{d}t = b\mu(A)$. Thus, $\mathbb{E}_\mu(XY) > \int_0^1 r_X(t) r_Y(t)\,\mathrm{d}t$ (resp. $\mathbb{E}_\mu(XY) < \int_0^1 r_X(t) r_Y(1-t)\,\mathrm{d}t$), which is a violation of the upper (resp. *lower*) bound in Theorem 2.1.

For an application of Theorem 2.1 to finance, the reader is referred to [3] (and the subsequent work [4]).

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