C. R. Acad. Sci. Paris, Ser. I 350 (2012) 1037–1042

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Partial Differential Equations/Probability Theory

Compensated fractional derivatives and stochastic evolution equations

Dérivées fractionnaires compensées et équations d'évolution stochastiques

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A R T I C L E   I N F O

Article history:
Received 6 August 2012
Accepted after revision 12 November 2012
Available online 24 November 2012

Presented by the Editorial Board

A B S T R A C T

We are interested in developing a pathwise theory for mild solutions of stochastic evolution equations when the noise path is β-Hölder continuous for $\beta \in (1/3, 1/2)$. From the point of view of the Rough Path Theory, stochastic integrals related to the solution of ordinary differential equations contain area-elements from a tensor space. Based on (compensated) fractional derivatives we are able to derive a second mild equation for these area components. We formulate sufficient conditions for the existence and uniqueness of a pathwise mild solution by using the Banach fixed point theorem provided that the coefficients of the system are sufficiently regular.

R É S U M É

Dans cette Note, nous sommes intéressés à développer une théorie trajectorielle pour les solutions 'mild' d'équations d'évolution stochastiques lorsque le bruit est β-Hölder continue pour $\beta \in (1/3, 1/2)$. Selon la théorie 'Rough Path', les intégrales stochastiques liés à la solution des équations différentielles ordinaires contiennent des éléments d'un espace de tenseurs. Grâce aux dérivées fractionnaires (compensées), on peut formuler une deuxième équation pour ce tenseur, pour lequel nous construisons un autre tenseur en fonction non seulement sur le bruit, mais aussi sur le semi-groupe. Nous formulons des conditions suffisantes pour l'existence et l'unicité d'une solution trajectorielle en utilisant le théoréme du point fixe de Banach lorsque des coefficients du système sont assez régulières.

Version française abrégée

La numérotation, les notations et la terminologie que nous utilisons ici se réfèrent directement aux formules, aux notations et à la terminologie de la version anglaise. Nous nous intéressons à l'existence et à l'unicité de la solution trajectorielle relative au problème (1). Plus précisément, nous avons le résultat suivant :

Théorème. *Supposons que* **H** *soit satisfaite. Si* $u_0 \in V_\delta$ *avec* $\delta \in (1 - \beta', 1]$, *alors le problème* (1) *possède une solution trajectorielle dans* $W(0, T)$.

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1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2012.11.007

Notre idée pour faire face à l'étude de solutions trajectorielles de (1) est de combiner la théorie 'Rough Path' avec des outils classiques, inspirés par le travail [5]. En fait, dans [5] les auteurs ont récemment étendu la méthode trajectorielle au cas d'un processus Hölderien avec un exposent de Hölder $H \in (1/3, 1/2]$. À cette fin, ils ont utilisé la soi-disant dérivée compensée fractionnaire, grâce à laquelle, ils ont été en mesure de formuler un résultat d'existence et d'unicité pour des équations différentielles stochastiques de dimension finie, ayant des coefficients suffisamment reguliers. Nous tenons à souligner que, pour formuler une équation d'opérateurs qui résoud ce problème, ils ont besoin d'une seconde équation dans l'espace des tenseurs.

Nous avons l'intention d'adapter les techniques de [5] pour obtenir des solutions trajectorielles pour notre équation d'évolution stochastique (1), en supposant que la partie linéaire génère un semi-groupe analytique sur un espace de Hilbert séparable. Cependant, il existe des différences significatives entre les deux travaux : dans le théorème d'existence, nous devons utiliser différentes techniques pour obtenir un point fixe de l'équation de l'opérateur. Nous serons en mesure d'obtenir des solutions dans de petits intervalles qui plus tard seront concaténés afin d'étendre la solution sur tout l'intervalle $[0, T]$. Il nous faut aussi trouver la bonne équation que va verifier le tenseur dans notre espace de dimension infinie. Pour l'obtenir, dans la première partie de cette Note nous considérons que notre équation d'évolution est perturbée par un bruit régulier, ce qui rend plus facile son analyse. A cet effet, il est d'une importance primordiale de construire un tenseur $\omega \otimes_S \omega$ en fonction du bruit ω ainsi que du semi-groupe S. Cette approche a l'avantage de rendre la définition de la solution plus claire, permettant de comprendre quelle est l'équation pour le tenseur associé à notre système. Le cas général nécessite l'introduction supplémentaire des approximations du bruit. Nous renvoyons le lecteur à la version anglaise pour une description plus complète de nos résultats, dont nous donnons les démonstrations détaillées dans [3].

1. Introduction, notation and fractional derivatives

Given a separable Hilbert space $(V, |\cdot|, (\cdot,\cdot))$, we consider partial differential equations (PDEs) driven by Hölder continuous functions with Hölder exponent $\beta \in (1/3, 1/2)$ of the type

$$\mathrm{d}u(t) = Au(t)\,\mathrm{d}t + G\big(u(t)\big)\,\mathrm{d}\omega(t), \qquad u(0) = u_0 \in V_\delta. \tag{1}$$

Here A is the infinitesimal generator of an analytic semigroup $S(\cdot)$ on V, and also a strictly negative operator generating a complete orthonormal basis $(e_i)_{i\in\mathbb{N}}$ in V with associated spectrum $(\lambda_i)_{i\in\mathbb{N}}$. By V_δ, $\delta \geqslant 0$, we denote the domain of $(-A)^\delta$ equipped with the norm $|x|_{V_\delta} = |(-A)^\delta x|$. Furthermore, G is a nonlinear term satisfying certain assumptions which will be described later and ω is the driven path. Note that as a particular case of driven noises we can consider a fractional Brownian motion (fBm) B^H with Hurst parameter $H \in (1/3, 1/2]$. Giving meaning to the definition of a solution for (1) is the main aim of this Note. For detailed proofs of the results presented here we refer to [3].

To the best of our knowledge there are only a few papers that use the Rough Path theory to solve stochastic differential equations and stochastic infinite-dimensional evolution equations. In [1] the characteristic method is used to study linear PDE driven by a nonregular term. Recently in [4] the authors have developed a theory of stochastic PDEs driven by rough paths that allows to treat semilinear problems, and it is based on a formalism which combines analytical semigroup theory and rough paths methods, where the nonlinearities are polynomial. A generalization of this paper is presented in [2], but they only can consider a finite-dimensional noisy input. In [8] the Szökefalvi–Nagy theorem is used to transform a mild equation into an infinite-dimensional ordinary differential equation for which the tools of Rough Paths theory can be used. On the other hand, the pathwise method has been extended to the case of dealing with a Hölder continuous driven process with Hölder exponent in $(1/3, 1/2)$ in [5]. Inspired by that work, we aim at studying mild solutions for the infinite-dimensional stochastic evolution equation (1) and to do that we combine the Fractional Calculus methods with classical tools for stochastic PDEs as an alternative to [1,2,4,8]. We emphasize that to formulate an operator equation solving this problem we need a second equation for the so-called *area* in the space of tensors. For this purpose it is of paramount importance to construct a tensor $\omega \otimes_S \omega$ depending on the noise path ω as well as on the semigroup S. This tensor will be rigorously defined below.

Given another separable Hilbert space $\tilde{V}$, denote by $L(V, \tilde{V})$ the Banach space of linear operators from V to $\tilde{V}$ and by $L_2(V, \tilde{V}) \subset L(V, \tilde{V})$ the space of Hilbert–Schmidt operators. $L_2(V \times V, \tilde{V})$ denotes the space of bilinear continuous mappings B from $V \times V$ which satisfy the Hilbert–Schmidt property. The topological tensor product of the Hilbert space V is denoted by $V \otimes V$ with norm $\|\cdot\|$. We remain that an operator $B \in L_2(V \times V, \tilde{V})$ can be extended by factorization to an operator $\hat{B} \in L_2(V \otimes V, \tilde{V})$, although we write for the extension $\hat{B}$ also the symbol B, see [6].

Let $0 \leqslant T_1 < T_2$. For $\beta \in (0,1)$ we introduce the space of β-Hölder continuous functions on $[T_1, T_2]$ with values in V, denoted by $C_\beta([T_1, T_2]; V)$, with the usual seminorm $\|\cdot\|_\beta$. Let Δ_{T_1,T_2} be the triangle $\{(s,t)\colon\ T_1 \leqslant s < t \leqslant T_2\}$. $C_{\beta+\beta'}(\Delta_{T_1,T_2}; V \otimes V)$ is the space of two-forms v with finite norm given by

$$\|v\|_{\beta+\beta'} = \sup_{s<t\in[T_1,T_2]} \frac{\|v(s,t)\|}{|t-s|^{\beta+\beta'}}, \quad \beta + \beta' < 1.$$

Consider $u \in C_\beta([T_1, T_2]; V)$, $\zeta \in C_{\beta'}([T_1, T_2]; V)$ and $v \in C_{\beta+\beta'}(\Delta_{T_1,T_2}; V \otimes V)$ such that the so called *Chen equality* holds, that is, for $T_1 \leqslant s \leqslant r \leqslant t \leqslant T_2$,

$$v(s,r)+v(r,t)+\big(u(r)-u(s)\big)\otimes_V\big(\zeta(t)-\zeta(r)\big)=v(s,t). \tag{2}$$

Now we aim at introducing the fractional derivatives and later at giving the pathwise interpretation of the stochastic integral. For $g,\ \zeta\in C_\gamma([T_1,T_2];\hat V)$, being $0<\alpha<\gamma<1$ and $\hat V$ some separable Hilbert space which will be given below, we define the fractional derivatives in the Weyl sense by

$$D^\alpha_{T_1+}g[r]=\frac{1}{\Gamma(1-\alpha)}\left(\frac{g(r)}{(r-T_1)^\alpha}+\alpha\int_{T_1}^{r}\frac{g(r)-g(q)}{(r-q)^{1+\alpha}}\,\mathrm{d}q\right)\in\hat V,$$

$$D^\alpha_{T_2-}\zeta_{T_2-}[r]=\frac{(-1)^\alpha}{\Gamma(1-\alpha)}\left(\frac{\zeta(r)-\zeta(T_2)}{(T_2-r)^\alpha}+\alpha\int_{r}^{T_2}\frac{\zeta(r)-\zeta(q)}{(q-r)^{1+\alpha}}\,\mathrm{d}q\right)\in\hat V,$$

where $T_1\leqslant r\leqslant T_2$, $\zeta_{T_2-}(r)=\zeta(r)-\zeta(T_2)$, and Γ denotes the Gamma function. If we assume that $g(T_1+)$, $\zeta(T_1+)$, $\zeta(T_2-)$ exist, being respectively the right side limit of g at T_1 and the right and left side limits of ζ at T_1, T_2, and $g_{T_1+}\in I^\alpha_{T_1+}(L^p((T_1,T_2);\mathbb{R}))$, $\zeta_{T_2-}\in I^\alpha_{T_2-}(L^q((T_1,T_2);\mathbb{R}))$ with $1/p+1/q\leqslant 1$ (see [9] for the definition of these spaces),

$$\int_{T_1}^{T_2}g\,\mathrm{d}\zeta=(-1)^\alpha\int_{T_1}^{T_2}D^\alpha_{T_1+}g_{T_1+}[r]D^{1-\alpha}_{T_2-}\zeta_{T_2-}[r]\,\mathrm{d}r+g(T_1+)\big(\zeta(T_2-)-\zeta(T_1+)\big).$$

Here $g_{T_1+}(\cdot)=g(\cdot)-g(T_1+)$. When ζ is not Lipschitz continuous we cannot define the above integral on the left-hand side in the classical way. Nevertheless, if $g\in C_\gamma([T_1,T_2];L_2(V,\tilde V))$, $\zeta\in C_\beta([T_1,T_2];V)$ for $0<\alpha<\gamma$, $1-\alpha<\beta$, then

$$\int_{T_1}^{T_2}g(r)\,\mathrm{d}\zeta(r)=(-1)^\alpha\int_{T_1}^{T_2}D^\alpha_{T_1+}g[r]D^{1-\alpha}_{T_2-}\zeta_{T_2-}[r]\,\mathrm{d}r,$$

which is well defined due to the separability of $\tilde V$, Pettis' theorem and because

$$\left|\int_{T_1}^{T_2}D^\alpha_{T_1+}g[r]D^{1-\alpha}_{T_2-}\zeta_{T_2-}[r]\,\mathrm{d}r\right|_{\tilde V}\leqslant c\|\zeta\|_\beta\big(\big\|g(T_1)\big\|_{L_2(V,\tilde V)}(T_2-T_1)^\beta+\|g\|_\gamma(T_2-T_1)^{\beta+\gamma}\big).$$

Indeed, if $\{\tilde e_i\}_{i\in\mathbb{N}}$ is a complete orthonormal basis of $\tilde V$, denote by π_m and $\tilde\pi_m$ the orthogonal projections on $\{e_1,\dots,e_m\}$ and $\{\tilde e_1,\dots,\tilde e_m\}$, resp., and define $\zeta_i=(\pi_i-\pi_{i-1})\zeta$, $g_{ji}=(\tilde\pi_j-\tilde\pi_{j-1})g(\pi_i-\pi_{i-1})$. Then we can express this Hilbert-space valued integral by the one dimensional integrals introduced above:

$$\int_{T_1}^{T_2}g(r)\,\mathrm{d}\zeta(r)=\sum_j\left(\sum_i\int_{T_1}^{T_2}D^\alpha_{T_1+}g_{ji}[r]D^{1-\alpha}_{T_2-}\zeta_{iT_2-}[r]\,\mathrm{d}r\right)\tilde e_j.$$

Thus, if $g(r)=G(u(r))$ and $u\in C_\gamma([T_1,T_2];V)$ for $\alpha<\gamma$, $\alpha+\beta>1$ and G has a bounded Fréchet derivative DG, then

$$\int_{T_1}^{T_2}G(u)\,\mathrm{d}\zeta=(-1)^\alpha\int_{T_1}^{T_2}D^\alpha_{T_1+}G\big(u(\cdot)\big)[r]D^{1-\alpha}_{T_2-}\zeta_{T_2-}[r]\,\mathrm{d}r. \tag{3}$$

If in addition G has a second bounded derivative we can rewrite the integral in (3) as follows

$$\int_{T_1}^{T_2}G(u)\,\mathrm{d}\zeta=(-1)^\alpha\int_{T_1}^{T_2}D^\alpha_{T_1+}\big(G\big(u(\cdot)\big)-DG\big(u(\cdot)\big)\big(u-u(T_1),\cdot\big)\big)[r]D^{1-\alpha}_{T_2-}\zeta_{T_2-}[r]\,\mathrm{d}r$$
$$+(-1)^\alpha\int_{T_1}^{T_2}D^\alpha_{T_1+}DG\big(u(\cdot)\big)\big(u-u(T_1),\cdot\big)[r]D^{1-\alpha}_{T_2-}\zeta_{T_2-}[r]\,\mathrm{d}r.$$

Suppose now that the above condition $\gamma>\alpha$ is not satisfied. Then $D^\alpha_{T_1+}G(u)$ is not well defined, in general. In this case it has sense to rewrite (3) by using the so called *compensated fractional derivative*

$$\hat{D}^{\alpha}_{T_1+}G\big(u(\cdot)\big)[r] = \frac{1}{\Gamma(1-\alpha)}\left(\frac{G(u(r))}{(r-T_1)^{\alpha}} + \alpha\int_{T_1}^{r}\frac{G(u(r))-G(u(q))-DG(u(q))(u(r)-u(q),\cdot)}{(r-q)^{\alpha+1}}\,\mathrm{d}q\right) \in L_2(V,\tilde{V}).$$

By making some computations, thanks to the fact that u, ξ and v are coupled by the Chen equality (2), we can express (3) under the weaker regularity condition $2\gamma > \alpha$ by

$$\int_{T_1}^{T_2} G(u)\,\mathrm{d}\zeta = (-1)^{\alpha}\int_{T_1}^{T_2}\hat{D}^{\alpha}_{T_1+}G\big(u(\cdot)\big)[r]D^{1-\alpha}_{T_2-}\zeta_{T_2-}[r]\,\mathrm{d}r - (-1)^{2\alpha-1}\int_{T_1}^{T_2}D^{2\alpha-1}_{T_1+}DG\big(u(\cdot)\big)[r]D^{1-\alpha}_{T_2-}\mathcal{D}^{1-\alpha}_{T_2-}v[r]\,\mathrm{d}r,$$

where $\mathcal{D}^{1-\alpha}_{T_2-}v$ for $v \in C_{\beta+\beta'}(\Delta_{T_1,T_2}; V\otimes V)$ denotes the following fractional derivative

$$\mathcal{D}^{1-\alpha}_{T_2-}v[r] = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)}\left(\frac{v(r,T_2)}{(T_2-r)^{1-\alpha}} + (1-\alpha)\int_{r}^{T_2}\frac{v(r,\tau)}{(\tau-r)^{2-\alpha}}\,\mathrm{d}\tau\right).$$

In what follows and till the end of this section we assume that $\omega : [0,T] \to V$ in (1) is smooth in the sense that ω is continuous at any t and continuously differentiable except at finitely many points. Assume $\tilde{V} = V_{\delta}$ for $\delta \in [0,1]$.

Lemma 1.1. *(See [7].) If G is Lipschitz continuous, the system (1) has a unique global solution which depends continuously on $u_0 \in V_{\delta}$. Moreover, $u \in C_{\beta}([0,T]; V)$ for $\beta \leqslant \delta \in [0,1)$.*

If in addition we assume that $G : V \to L_2(V, V_{\delta})$ is twice continuously Fréchet–differentiable with bounded second derivative, following the steps in the proof of Theorem 3.3 in [5], we can rewrite (1) as

$$\begin{aligned} u(t) = {} & S(t)u_0 + (-1)^{\alpha}\int_0^t \hat{D}^{\alpha}_{0+}\big(S(t-\cdot)G\big(u(\cdot)\big)\big)[r]D^{1-\alpha}_{t-}\omega_{t-}[r]\,\mathrm{d}r \\ & - (-1)^{2\alpha-1}\int_0^t D^{2\alpha-1}_{0+}\big(S(t-\cdot)DG\big(u(\cdot)\big)\big)[r]D^{1-\alpha}_{t-}\mathcal{D}^{1-\alpha}_{t-}(u\otimes\omega)[r]\,\mathrm{d}r. \end{aligned} \tag{4}$$

Operators G of this kind can be represented by integral operators with sufficiently regular nonlinear kernel for an appropriate Hilbert space V. Note that when ω is regular we can take $v = (u\otimes\omega)$ which can be defined by the integral $(u\otimes\omega)(s,t) = \int_s^t (u(r)-u(s))\otimes_V \omega'(r)\,\mathrm{d}r$, but in the non-smooth case this is not possible, hence we need a second equation to define it. In fact, to obtain an adequate expression of $(u\otimes\omega)$ we need to introduce a tensor which also involves the semigroup: for $\beta' \in (1/3, 1/2)$ and $\alpha \in (0,1)$ we define $\omega_S : \Delta_{0,T} \to L_2(V_{\delta}, V\otimes V)$ and then $(\omega\otimes_S\omega) : \Delta_{0,T}\times L_2(V,V_{\delta}) \to V\otimes V$ by

$$\omega_S(s,t)\cdot = (-1)^{-\alpha}\int_s^t \big(S(\xi-s)\cdot\big)\otimes_V \omega'(\xi)\,\mathrm{d}\xi, \qquad E(\omega\otimes_S\omega)(s,t) = (-1)^{\alpha}\int_s^t \omega_S(r,t)E\big(\omega'(r)\big)\,\mathrm{d}r, \tag{5}$$

for $0 \leqslant s \leqslant t \leqslant T$. Under the condition

$$\sum_{i=1}^{\infty}\lambda_i^{-1-2\delta} < \infty \tag{6}$$

$(\omega\otimes_S\omega)$ is in $C_{2\beta'}(\Delta_{0,T}; L_2(L_2(V,V_{\delta}), V\otimes V))$. Moreover, the following lemma plays an essential role in our results:

Lemma 1.2. *Suppose that (6) holds. For $0 \leqslant s \leqslant t \leqslant T$, $(u\otimes\omega)$ satisfies the equation*

$$\begin{aligned} (u\otimes\omega)(s,t) = {} & \int_s^t \big(S(\xi-s)-\mathrm{id}\big)u(s)\otimes_V \omega'(\xi)\,\mathrm{d}\xi - (-1)^{\alpha}\int_s^t \hat{D}^{\alpha}_{s+}G\big(u(\cdot)\big)[r]D^{1-\alpha}_{t-}(\omega\otimes_S\omega)(\cdot,t)_{t-}[r]\,\mathrm{d}r \\ & + (-1)^{2\alpha-1}\int_s^t D^{2\alpha-1}_{s+}DG\big(u(\cdot)\big)[r]D^{1-\alpha}_{t-}\mathcal{D}^{1-\alpha}_{t-}\big(u\otimes(\omega\otimes_S\omega)(t)\big)[r]\,\mathrm{d}r, \end{aligned}$$

where for $\tilde{E} \in L_2(V\otimes V, V_{\delta})$ and $0 \leqslant s \leqslant q \leqslant t \leqslant T$,

$$\tilde{E}\big(u\otimes(\omega\otimes_S\omega)(t)\big)(s,q)=(-1)^{\alpha-1}\int_s^q D_{s+}^{2\alpha-1}\tilde{E}\big(u(\cdot)-u(s),\cdot\big)[r]D_{q-}^{1-\alpha}\mathcal{D}_{q-}^{1-\alpha}\big(\omega_S(t)\otimes\omega\big)[r]\,\mathrm{d}r$$
$$-\int_s^q \hat{D}_{s+}^{\alpha}\omega_S(\cdot,t)\tilde{E}\big(u(\cdot)-u(s),\cdot\big)[r]D_{q-}^{1-\alpha}\omega_{q-}[r]\,\mathrm{d}r$$
$$+(-1)^{\alpha-1}\int_s^q D_{s+}^{2\alpha-1}\omega_S(\cdot,t)[r]\tilde{E}D_{q-}^{1-\alpha}\mathcal{D}_{q-}^{1-\alpha}(u\otimes\omega)(t)[r]\,\mathrm{d}r, \tag{7}$$

where $(\omega_S(t)\otimes\omega)$ *is defined for* $s\leqslant\tau\leqslant t$, $E\in L_2(V,V_\delta)$ *as*

$$E\big(\omega_S(t)\otimes\omega\big)(s,\tau)=\omega_S(\tau,t)\int_s^\tau S(\tau-r)E\,\mathrm{d}\omega(r)+E(\omega\otimes_S\omega)(s,\tau)-\omega_S(s,t)E\big(\omega(\tau)-\omega(s)\big).$$

2. Existence and uniqueness of pathwise mild solutions

Assume that ω is a Hölder continuous function of order $\beta'\in(1/3,1/2)$, hence the integral with integrator ω is not well defined in the classical sense. However, in what follows we will see that the two last terms in (4) are well defined. A main difficulty in that point is that for a nonregular path ω we cannot expect in general that $(u\otimes\omega)$, which appears in the last term on (4), is well defined. However, we are able to overcome these problems by formulating the term $(u\otimes\omega)$ by another operator equation. This will be possible thanks to the $2\beta'$-Hölder continuity of $(\omega\otimes_S\omega)$.

For every β'-Hölder continuous path ω we take as the phase space the following one: for $0\leqslant T_1<T_2$, $W(T_1,T_2)=C_\beta([T_1,T_2];V)\times C_{\beta+\beta'}(\Delta_{T_1,T_2};V\otimes V)$, with seminorm $|||U|||=\|u\|_\beta+\|v\|_{\beta+\beta'}$, for $U=(u,v)\in W(T_1,T_2)$, and such that the Chen equality holds for U, which means that for $T_1\leqslant s\leqslant r\leqslant t\leqslant T_2$,

$$v(s,r)+v(r,t)+\big(u(r)-u(s)\big)\otimes_V\big(\omega(t)-\omega(r)\big)=v(s,t),$$

where ω denotes a fixed β'-Hölder path with $\beta'\in(1/3,1/2)$. We now give the definition of a solution to (1).

Definition 2.1. Let $T>0$. Assume that $u_0\in V$, and G satisfies the conditions described on Section 1. A mild solution of (1) is a pair $U=(u,v)\in W(0,T)$ satisfying

$$u(t)=S(t)u_0+(-1)^\alpha\int_0^t \hat{D}_{0+}^{\alpha}\big(S(t-\cdot)G\big(u(\cdot)\big)\big)[r]D_{t-}^{1-\alpha}\omega_{t-}[r]\,\mathrm{d}r$$
$$-(-1)^{2\alpha-1}\int_0^t D_{0+}^{2\alpha-1}\big(S(t-\cdot)DG\big(u(\cdot)\big)\big)[r]D_{t-}^{1-\alpha}\mathcal{D}_{t-}^{1-\alpha}v[r]\,\mathrm{d}r, \tag{8}$$

$$v(s,t)=\int_s^t\big(S(\xi-s)-\mathrm{id}\big)u(s)\otimes_V\,\mathrm{d}\omega(\xi)-(-1)^\alpha\int_s^t \hat{D}_{s+}^{\alpha}G\big(u(\cdot)\big)[r]D_{t-}^{1-\alpha}(\omega\otimes_S\omega)(\cdot,t)_{t-}[r]\,\mathrm{d}r$$
$$+(-1)^{2\alpha-1}\int_s^t D_{s+}^{2\alpha-1}DG\big(u(\cdot)\big)[r]D_{t-}^{1-\alpha}\mathcal{D}_{t-}^{1-\alpha}\big(u\otimes(\omega\otimes_S\omega)(t)\big)[r]\,\mathrm{d}r, \tag{9}$$

for $0\leqslant s<t\leqslant T$. The term $(u\otimes(\omega\otimes_S\omega)(t))(s,t)$ is defined by the right-hand side of (7) replacing $u\otimes\omega$ by v.

To establish the existence and uniqueness of solutions to (8)–(9), we set the following Hypothesis **H**:

(1) Assume that $\sum_i\lambda_i^{-2\delta}<\infty$, where $\delta\in(1-\beta',1]$. Assume $G:V\mapsto L_2(V,V_\delta)$ is a three times Fréchet differentiable mapping with bounded derivatives, such that $DG(\cdot)\in L_2(V\otimes V,V_\delta)$.
(2) Suppose that $1/3<H\leqslant 1/2$ and $1/3<\beta<H$, and that there is an α with $1-\beta<\alpha<2\beta$, $\alpha<\frac{\beta+1}{2}$ and $-3\beta+\alpha+H>0$; $\omega\in C_{\beta'}([0,T];V)$ for any $\beta<\beta'<H$ with the last two inequalities hold replacing H by β'.
(3) Assume $(\omega\otimes_S\omega)\in C_{2\beta'}(\Delta_{0,T};L_2(L_2(V,V_\delta),V\otimes V))$.
(4) Let $(\omega^n)_{n\in\mathbb{N}}$ be a sequence of piecewise smooth functions with values in V such that $((\omega^n\otimes_S\omega^n))_{n\in\mathbb{N}}$ is defined by (5). Assume then that for any $\beta'<H$ the sequence $((\omega^n,(\omega^n\otimes_S\omega^n(\cdot,\cdot))))_{n\in\mathbb{N}}$ converges to $(\omega,(\omega\otimes_S\omega(\cdot,\cdot)))$ in $C_{\beta'}([0,T];V)\times C_{2\beta'}(\Delta_{0,T};L_2(L_2(V,V_\delta),V\otimes V))$.

For $0 < a < b$ we consider the *concatenation* of elements of $W(0,a)$ and $W(a,b)$, taking into account that the elements of these function spaces consist of a path component and an area component, and that the concatenation of the latter should be done in agreement with the Chen equality. Therefore, given $U^1 = (u^1, v^1) \in W(0,a)$ such that $u^1(0) \in V_\delta$, for $\delta \in [0,1]$ and $U^2 = (u^2, v^2) \in W(a,b)$ such that $u^2(a) = u^1(a)$, for $0 \leqslant a < b \leqslant 1$, we define $U = (u, v)$ as follows:

$$u(t) = \begin{cases} u^1(t): & 0 \leqslant t \leqslant a, \\ u^2(t): & a \leqslant t \leqslant b, \end{cases} \qquad v(s,t) = \begin{cases} v^1(s,t): & 0 \leqslant s \leqslant t \leqslant a, \\ v^2(s,t): & a \leqslant s \leqslant t \leqslant b, \\ (u^1(a) - u^1(s)) \otimes_V (\omega(t) - \omega(a)) + v^1(s,a) + v^2(a,t): & s \leqslant a < t. \end{cases}$$

If $\omega \in C_{\beta'}([0,b]; V)$ the concatenation U of U^1, U^2 is in $W(0,b)$. The main theorem of this Note is the following:

Theorem 2.2. *Under Hypothesis* **H**, *if* $u_0 \in V_\delta$ *with* $\delta \in (1-\beta', 1]$, *there exists a unique mild solution to system* (8)–(9) *in* $W(0,T)$.

This result can be proven thanks to the Banach fixed point theorem. It is interesting to stress that because the coefficients of this system are not Lipschitz we need to split the time interval into a finite sequence of subintervals on which the Banach fixed point theorem can be applied, and finally the solutions on each subinterval have to be *glued* by using the above concatenation property. We refer to [3] for more details and for the proofs of the above results.

Acknowledgements

This work was partially supported by MTM 2011-22411 (M.J. Garrido-Atienza and B. Schmalfuß) and a Distinguished Professorship at BYU/Utah (B. Schmalfuß).

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