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Differential Geometry

Zero mean curvature surfaces in $\mathbf{L}^3$ containing a light-like line

Surfaces de courbure moyenne nulle dans $\mathbf{L}^3$ contenant des droites de type lumière

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ABSTRACT

It is well known that space-like maximal surfaces and time-like minimal surfaces in Lorentz–Minkowski 3-space $\mathbf{L}^3$ have singularities (i.e. points where the induced metric degenerates) in general. We are interested in the case where the singular set consists of a light-like line, since this case has not been analyzed before. In this Note, we give new examples of such surfaces.

RÉSUMÉ

Il est bien connu que les surfaces maximales de type espace et les surfaces minimales de type temps dans l'espace $\mathbf{L}^3$ de Lorentz–Minkowski de dimension 3 possèdent en général des singularités. Ces deux types sont caracterisés comme des surfaces de courbure moyenne nulle. La Note considère le cas où le lieu des singularités consiste en une droite de type lumière, cette situation n'ayant semble-t-il pas encore été analysée. Dans cette Note, nous donnons de nouveaux exemples de telles surfaces.

1. Introduction

We denote by $(\mathbf{L}^3; t, x, y)$ the Lorentz–Minkowski 3-space of signature $(-++)$. Many examples of space-like maximal surfaces in $\mathbf{L}^3$ containing singular curves have been constructed (cf. [10,1,6] and [11,3]). In particular, if one gives a generic regular curve γ in $\mathbf{L}^3$ whose velocity vector field is light-like, then there exists a zero mean curvature surface which changes its causal type across this curve from a space-like maximal surface to a time-like minimal surface (cf. [4,9,7] and [8]). This construction can be accomplished using the Björling formula for the Weierstrass-type representation formula of maximal

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surfaces. However, if γ is a light-like line, the construction fails, since the isothermal coordinates break down along the light-like singular points. Locally, such surfaces are the graph of a function $t = f(x, y)$ satisfying

$$\left(1 - f_y^2\right) f_{xx} + 2 f_x f_y f_{xy} + \left(1 - f_x^2\right) f_{yy} = 0. \tag{1}$$

We call this and its graph the *zero mean curvature equation* and *zero mean curvature surface*, respectively. Gu [4] and Klyachin [9] gave several fundamental results on zero mean curvature surfaces which might change type, but the surfaces containing a singular light-like line have not yet been sufficiently discussed. Moreover, Klyachin pointed out the importance of this case in [9, Theorem 1]. Here, we consider such surfaces and construct new examples.

2. Characteristic of zero mean curvature surfaces along a singular light-like line

Suppose that a C^∞-function $t = f(x, y)$ is a solution of the zero mean curvature equation (1) and that its graph contains a singular light-like line (segment) L. By a suitable motion, we may assume that L is included in $\{(t, 0, t) \in \mathbf{L}^3;\ t \in \mathbf{R}\}$ and that, locally,

$$f(x, y) = y + \frac{\alpha(y)}{2} x^2 + \beta(x, y) x^3, \tag{2}$$

where $\alpha(y)$ and $\beta(x, y)$ are C^∞-functions. We define

$$A := \left(1 - f_y^2\right) f_{xx} + 2 f_x f_y f_{xy} + \left(1 - f_x^2\right) f_{yy}, \qquad B := 1 - f_x^2 - f_y^2. \tag{3}$$

Note that $B > 0$ (resp. $B < 0$) if and only if the graph is space-like (resp. time-like). By direct calculations with (2), we see the following, where $\alpha' := d\alpha/dy$, $\alpha'' := d^2\alpha/dy^2$,

$$A|_{x=0} = A_x|_{x=0} = 0, \qquad A_{xx}|_{x=0} = 2\alpha\alpha' + \alpha'', \tag{4}$$

$$B|_{x=0} = B_x|_{x=0} = 0, \qquad B_{xx}|_{x=0} = -2\left(\alpha' + \alpha^2\right). \tag{5}$$

Since A must vanish identically, we have $A_{xx}|_{x=0} = 0$, hence there exists a constant $\mu \in \mathbf{R}$ such that

$$\alpha' + \alpha^2 + \mu = 0. \tag{6}$$

We call μ the *characteristic* of the zero mean curvature surface f along L. Then $B_{xx}|_{x=0} = 2\mu$ and the following assertion is immediate:

Proposition 1. *If $\mu > 0$ (resp. $\mu < 0$), then the graph of $t = f(x, y)$ is space-like (resp. time-like) on both sides of L.*

In particular, the graph might change type across L from space-like to time-like only if the characteristic μ vanishes. However, even in this case, the graph might not change type (cf. Examples 1 and 2).

By a homothetic change of the graph $f_c(x, y) := f(cx, cy)/c$ $(c > 0)$, one can normalize the characteristic μ to be $-1, 0, 1$. For these three cases, we have the following general solutions to (6):

$$\mu = 1: \quad \alpha^+ := -\tan(y + C) \quad (C \in \mathbf{R}),$$

$$\mu = 0: \quad \alpha_I^0 := 0 \quad \text{or} \quad \alpha_{II}^0 := (y + C)^{-1} \quad (C \in \mathbf{R}),$$

$$\mu = -1: \quad \alpha_I^- := \tanh(y + C), \qquad \alpha_{II}^- := \coth(y + C), \qquad \alpha_{III}^- := 1 \text{ or } -1 \quad (C \in \mathbf{R}).$$

Theorem 2. *For each choice of $\alpha = \alpha^+, \alpha_I^0, \alpha_{II}^0, \alpha_I^-, \alpha_{II}^-$ or α_{III}^-, there exists a real analytic zero mean curvature surface in $\mathbf{L}^3$ containing a light-like line (segment) which attains α.*

We prove this theorem in the next section by showing explicit examples.

3. Examples of zero mean curvature surfaces

Example 1. (See Klyachin [9].) Let $f(x, y) = y + g(x)$, where $g(x)$ is a C^∞-function. Then f is a solution of (1), whose graph has a singular light-like line if and only if $g(0) = dg/dx(0) = 0$. The surface does not change type and $\mu = 0$, since $B = -g'(x)^2 \leqslant 0$. When g vanishes, f gives a light-like plane and $\alpha = \alpha_I^0$.

Example 2 *(Hyperbolic catenoids).* The subset of $\mathbf{L}^3$ given by

$$\mathcal{S} = \left\{(t, x, y) \in \mathbf{L}^3;\ \sin^2 x + y^2 - t^2 = 0\right\} \tag{7}$$

gives a singly-periodic space-like maximal surface with conical singularities at $(0, n\pi, 0)$ $(n \in \mathbf{Z})$. This surface contains the light-like lines $\{(t, x, y) \in \mathbf{L}^3;\ t = \pm y,\ x \in \pi\mathbf{Z}\}$, where the first fundamental form degenerates, see Fig. 1(a). It is well known

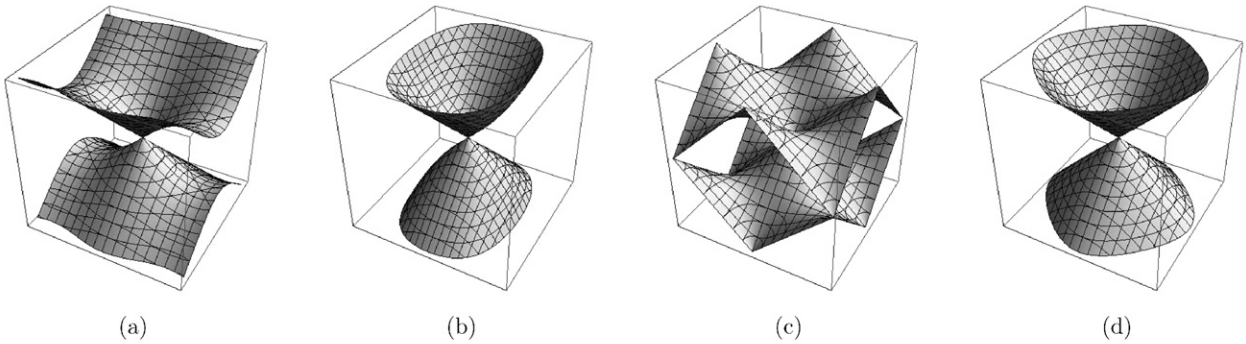

Fig. 1. Space-like hyperbolic catenoid (a), time-like hyperbolic catenoid (b), space-like Scherk surface (c), and time-like Scherk surface of first kind (d). Space-like maximal surfaces and time-like minimal surfaces have their conjugate maximal surfaces (cf. [10]) and conjugate minimal surfaces (cf. [5]), respectively. In this correspondence, the cone-like singularities of the surfaces (a)–(d) correspond to fold singularities, and it is known that space-like maximal surfaces can be extended real-analytically to time-like minimal surfaces along fold singularities (cf. [4,8]). In fact, the conjugate surfaces of (a) and (b) are the space-like hyperbolic helicoid and the time-like hyperbolic helicoid (cf. [8, Lemma 2.11]), both of which are subsets of the entire graph $t = y\tanh x$ given in [10, Prop. 5.1(i)]. Also both the conjugate surfaces of (c) and (d) are subsets of the entire graph $t = \log(\cosh y/\cosh x)$ given in Eq. (9). See [2] for details.

that the space-like hyperbolic catenoid is given by

$$\varphi_1(u,v) := (\cosh u \sin u, v, \sinh u \sin v) \quad (u, v \in \mathbf{R}).$$

However, this parametrization covers only half of $\mathcal{S}$. This duplication (7) of the hyperbolic catenoid as an analytic continuation was discovered by Klyachin [9].

On the other hand, the time-like hyperbolic catenoid is classically known as well, which is given by

$$\varphi_2(u,v) := \frac{1}{2}(\sinh u + \sinh v, u+v, \cosh u - \cosh v) \quad (u, v \in \mathbf{R}).$$

The image of φ_2 can be extended analytically to the time-like minimal surface

$$\mathcal{T} = \{(t,x,y) \in \mathbf{L}^3;\ \sinh^2 x + y^2 - t^2 = 0\}, \tag{8}$$

which is immersed except at the origin (see Fig. 1(b)). This extendability is the same phenomenon as for the space-like hyperbolic catenoid $\mathcal{S}$, but does not seem to have been pointed out before in the literature. Note that the original catenoid in Euclidean 3-space $(\mathbf{R}^3; x, y, z)$ has the expression $\cosh^2 z - x^2 - y^2 = 0$. $\mathcal{T}$ contains two light-like lines where the first fundamental form degenerates. Both $\mathcal{S}$ and $\mathcal{T}$ do not change type across the singular lines and have $\mu = 0$ and $\alpha = \alpha_I^0 (= 0)$.

Since the light cone $t = \pm\sqrt{x^2 + y^2}$ gives a trivial zero mean curvature surface with $\alpha = \alpha_{II}^0$, it is now sufficient to find surfaces for $\alpha = \alpha^+, \alpha_I^-, \alpha_{II}^-$ and α_{III}^- to prove Theorem 2. Osamu Kobayashi [10, Prop.5.1(ii)] pointed out the existence of a zero mean curvature surface given by the entire graph

$$e^t \cosh x = \cosh y, \tag{9}$$

which changes type, as an analogue of the classical minimal surface $e^z \cos x = \cos y$, called Scherk's surface (see also [8]). Although (9) does not contain a light-like line, the authors discovered all the following examples as variants of (9), all of which contain light-like lines (see also the caption of Fig. 1).

Example 3 *(Space-like Scherk surface).* The surface in $\mathbf{L}^3$ given by $\cos t = \cos x \cos y$ is triply periodic, admits only cone-like singular points, and has $\mu = 1$ with $\alpha = \alpha^+$ (see Fig. 1(c)).

Example 4 *(Time-like Scherk surface of first kind).* The surface in $\mathbf{L}^3$ given by $\cosh t = \cosh x \cosh y$ is immersed everywhere except one cone-like singular point, having $\mu = -1$ with $\alpha = \alpha_I^-$ (see Fig. 1(d)).

Example 5 *(Time-like Scherk surface of second kind).* The surface in $\mathbf{L}^3$ given by $\sinh t = \cosh x \sinh y$ is an immersed surface with $\mu = -1$ and $\alpha = \alpha_{II}^-$, which is an entire graph over the xy-plane. Taking a limit of $\cosh x = \sinh(t+\delta)/\sinh(y+\delta)$ as $\delta \to \pm\infty$, we get a pair of degenerate time-like Scherk surfaces $t = y \pm \log(\cosh x)$ as special cases of Example 1 which is a graph with characteristic $\mu = -1$ and $\alpha = \alpha_{III}^-$. The proof of Theorem 2 is now complete.

All the aforementioned zero mean curvature surfaces containing a light-like line L do not change type across L, and it is interesting to ask whether there exist zero mean curvature surfaces which change type across L. We announce the following existence result for solutions of Eq. (1), which is proved by solving the sequence of ordinary differential equations $\partial^n A/\partial x^n(0) = 0$ for $n \geqslant 3$:

Theorem 3. *(See [2].) There exists a family of real analytic zero mean curvature surfaces which contain a light-like line segment L, change type across L, and have neither even nor odd symmetry with respect to L.* (*Note that all the surfaces in Examples* 2–5 *satisfy the property* $f(-x, y) = f(x, y)$.)

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