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Model spaces for sharp isoperimetric inequalities ☆

Espaces modèles pour des inégalités isopérimétriques optimales

Emanuel Milman

Department of Mathematics, Technion – Israel Institute of Technology, Haifa 32000, Israel

A R T I C L E   I N F O

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A B S T R A C T

We obtain new sharp isoperimetric inequalities on a Riemannian manifold equipped with a probability measure, whose generalized Ricci curvature is bounded from below (possibly negatively), and generalized dimension and diameter of the convex support are bounded from above (possibly infinitely). Our inequalities are *sharp* for sets of any given measure and with respect to all parameters (curvature, dimension and diameter). Moreover, for each choice of parameters, we identify the *model spaces* which are extremal for the isoperimetric problem. In particular, we recover the Gromov–Lévy and Bakry–Ledoux isoperimetric inequalities, which state that whenever the curvature is strictly *positively* bounded from below, these model spaces are the n-sphere and Gauss space, corresponding to generalized dimension being n and ∞, respectively. In all other cases, which seem new even for the classical Riemannian-volume measure, it turns out that there is no *single* model space to compare to, and that a simultaneous comparison to a natural *one parameter family* of model spaces is required, nevertheless yielding a sharp result.

R É S U M É

Nous obtenons de nouvelles inégalités isopérimétriques optimales sur une variété Riemannienne munie d'une mesure de probabilité, dont la courbure de Ricci généraliée (pouvant prendre des valeurs négatives) est bornée inférieurement, et dont la dimension généralisée et le diamètre du support convexe sont bornés supérieurement (éventuellement infinis). Nos inégalités sont *optimales* pour les ensembles de mesure fixée et par rapport à tous les paramètres (courbure, dimension et diamètre). De plus, pour tout choix des paramètres, nous identifions les *espaces modèles* qui sont extrémaux pour le problème isopérimétrique considéré. En particulier, nous retrouvons les inégalités isopérimétriques de Gromov–Lévy et de Bakry–Ledoux, qui montrent que lorsque la courbure est bornée inférieurement par une constante strictement positive, ces modèles sont la sphère de dimension n (lorsque la dimension généralisée est n) et l'espace de Gauss (lorsque la dimension généralisée est infinie). Dans tous les autres cas, notre résultat semble nouveau même dans le cas classique de la mesure de volume, et montre qu'en réalité il n'y a pas d'*unique* espace modèle, mais que cependant une comparaison simultanée avec une famille naturelle *à un paramètre* d'espaces modèles est nécessaire et fournit un résultat optimal.

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E-mail address: emilman@tx.technion.ac.il.

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Soit (M^n, g) une variété Riemannienne orientée, lisse, de dimension $n \geqslant 2$, et soit μ une mesure de probabilité sur M ayant une densité Ψ par rapport à la mesure Riemanienne vol_g.

Définition 1 *(Condition de Courbure-Dimension-Diamètre).* On dit que (M^n, g, μ) possède la condition de Courbure-Dimension-Diamètre $CDD(\rho, n+q, D)$ ($\rho \in \mathbb{R}$, $q \in [0, \infty]$, $D \in (0, \infty]$), si μ est supportée dans la clôture d'un domaine géodésiquement convexe $\Omega \subset M$ de diamètre au plus D, ayant un bord C^2 (pouvant être vide), et si, écrivant $\mu = \Psi \cdot vol_g|_\Omega$, on a $\Psi > 0$ sur $\overline{\Omega}$ et $\log(\Psi) \in C^2(\overline{\Omega})$, et si l'égalité suivant entre 2-tenseurs est vérifiée :

$$Ric_g - \nabla_g^2 \log(\Psi) - \frac{1}{q} \nabla_g \log(\Psi) \otimes \nabla_g \log(\Psi) = Ric_g - q \frac{\nabla_g^2 \Psi^{1/q}}{\Psi^{1/q}} \geqslant \rho g \quad \text{sur } \Omega. \tag{1}$$

Lorsque $\Omega = M$ et $D = +\infty$, la définition précédente coïncide avec la condition de Courbure-Dimension $CD(\rho, n+q)$, introduite dans une forme équivalente par Bakry et Émery dans [2]. En effet, le tenseur de Ricci généralisé dans (1) inclus une information sur la courbure et la dimension à partir de la géométrie de (M, g) et de la mesure μ, et donc ρ peut être considéré comme une borne inférieure de la courbure généralisée, et $n+q$ comme la borne supérieure de la dimension généralisée.

Lorsque $\rho > 0$, les inégalités isopérimétriques optimales sous la condition $CD(\rho, n+q)$ sont connues et bien comprises, grâce à l'existence d'espaces modèles de comparaison sur lesquels l'égalité est atteinte. Le premier résultat de ce type a été obtenu par M. Gromov dans [12] (voir [13, Appendix C]), qui identifia la sphère de dimension n comme l'espace modèle extrémal dans le cas d'une densité constante ($q = 0$), étendant ainsi l'inégalité isopérimétrique de P. Lévy à la sphère [15,23]. Le cas $q = +\infty$ a été traité par Bakry et Ledoux [3] (cf. Morgan [20]), qui ont montrés que l'espace modèle correspondant et la droite réelle munie de la densité gaussienne. Une extension de ces résultats au cas $q \in (0, \infty)$ a été plus tard obtenue par Bayle dans [6, Appendix E].

Lorsque $\rho \leqslant 0$, la situation est très différente, et sans imposer de condition *supplémentaire* sur l'espace (M, g, μ), aucune inégalité isopérimétrique ne peut être déduite de la condition $CD(\rho, n+q)$ sans hypothèse supplémentaire, et donc la condition $CDD(\rho, n+q, D)$ est une amélioration naturelle. Cependant, aucun des résultats connus lorsque $\rho \leqslant 0$ ne fournit des ingalités isopérimétriques *optimales* (cf. [8,7,10]). La difficulté lorsque $\rho \leqslant 0$ se trouve dans le fait qu'il ne semble pas y avoir un *bon* espace modèle de comparaison, comme dans les résultats de Gromov–Lévy ou Bakry–Ledoux. Le but de cet article est de combler ce manque, en fournissant une inégalité isopérimétrique optimale sous la condition $CDD(\rho, n+q, D)$ pour tout $\rho \in \mathbb{R}$, $q \in [0, \infty]$, $D \in (0, \infty]$, dans un cadre unique et unifié. En particuler, pour tout choix des paramètres, nous identifions les *espaces modèles* qui sont extrémaux pour le problème isopérimétrique ; il se trouve qu'hormis les cas traités par Gromov–Lévy et Bakry–Ledoux, il n'y a pas d'*unique* espace modèle de comparaison, et qu'une comparaison simultanée avec une famille naturelle à un paramètre d'espaces modèles est nécessaire, et fournit un résultat de comparaison optimal (voir le Théorème 2 et le Corollaire 3). Nos résultats semblent nouveaux même dans le cas classique où la densité est constante ($q = 0$) lorsque $\rho \leqslant 0$ et $D < \infty$ ou lorsque $\rho > 0$ et $D < \pi\sqrt{(n-1)/\rho}$.

Notre méthode de preuve est entièrement géométrique, suivant l'approche développée par Gromov dans [12]. Nous utilisons de manière cruciale des résultats de Théorie Géométrique de la Mesure concernant la régularité des minimiseurs isopérimétriques, à la fois à l'intérieur et sur le bord. Pour estimer la mesure balayée par l'application normale émanant de la partie régulière du minimiseur sous la condition $CDD(\rho, n+q, D)$, nous utilisons une version généralisée du théorème de Heintze–Karcher due à V. Bayle [6, Appendix E] et F. Morgan [20]. Pour prouver l'optimalité des résultats, nous implémentons nos densités modèles de dimension 1 sur un domaine géodésiquement convexe d'une variété de dimension n, en les épaississant légèrement en dimension $n-1$.

1. Introduction

Let (M^n, g) denote an n-dimensional ($n \geqslant 2$) complete oriented smooth Riemannian manifold, and let μ denote a probability measure on M having density Ψ with respect to the Riemannian volume form vol_g.

Definition 1 *(Generalized Ricci Tensor).* Given $q \in [0, \infty]$ and assuming that $\Psi > 0$ and $\log(\Psi) \in C^2$, we denote by $Ric_{g,\Psi,q}$ the following generalized Ricci tensor:

$$Ric_{g,\Psi,q} := Ric_g - \nabla_g^2 \log(\Psi) - \frac{1}{q} \nabla_g \log(\Psi) \otimes \nabla_g \log(\Psi) = Ric_g - q \frac{\nabla_g^2 \Psi^{1/q}}{\Psi^{1/q}}. \tag{2}$$

Note that $Ric_{g,\Psi,\infty} = Ric_g - \nabla_g^2 \log(\Psi)$ and that $Ric_{g,\Psi,0} = Ric_g$ when Ψ is constant. Here as usual Ric_g denotes the Ricci curvature tensor and ∇_g denotes the Levi-Civita covariant derivative.

Definition 2 *(Curvature-Dimension-Diameter Condition).* (M^n, g, μ) is said to satisfy the Curvature-Dimension-Diameter Condition $CDD(\rho, n+q, D)$ ($\rho \in \mathbb{R}$, $q \in [0, \infty]$, $D \in (0, \infty]$), if μ is supported on the closure of a geodesically convex domain

$\Omega \subset M$ of diameter at most D, having (possibly empty) C^2 boundary, $\mu = \Psi \cdot vol_g|_\Omega$ with $\Psi > 0$ on $\overline{\Omega}$ and $\log(\Psi) \in C^2(\overline{\Omega})$, and as 2-tensor fields:

$$Ric_{g,\Psi,q} \geqslant \rho g \quad \text{on } \Omega.$$

When $\Omega = M$ and $D = +\infty$, the latter definition coincides with the celebrated Bakry–Émery Curvature-Dimension Condition $CD(\rho, n+q)$, introduced in an equivalent form in [2] (in the more abstract framework of diffusion generators). Indeed, the generalized Ricci tensor incorporates information on curvature and dimension from both the geometry of (M, g) and the measure μ, and so ρ may be thought of as a generalized-curvature lower bound, and $n+q$ as a generalized-dimension upper bound. The generalized Ricci tensor (2) was introduced with $q = \infty$ in [16,17] and in general in [1] (cf. [18]), and has been extensively studied and used in recent years (see e.g. also [22,14,25,21,5,24,19,26] and the references therein).

Let (Ω, d) denote a separable metric space, and let μ denote a Borel probability measure on (Ω, d). The Minkowski (exterior) boundary measure $\mu^+(A)$ of a Borel set $A \subset \Omega$ is defined as $\mu^+(A) := \liminf_{\varepsilon \to 0} \frac{\mu(A^d_\varepsilon) - \mu(A)}{\varepsilon}$, where $A_\varepsilon = A^d_\varepsilon := \{x \in \Omega;\ \exists y \in A,\ d(x, y) < \varepsilon\}$ denotes the ε extension of A with respect to the metric d. The isoperimetric profile $\mathcal{I} = \mathcal{I}(\Omega, d, \mu)$ is defined as the pointwise maximal function $\mathcal{I} : [0,1] \to \mathbb{R}_+ \cup \{+\infty\}$, so that $\mu^+(A) \geqslant \mathcal{I}(\mu(A))$, for all Borel sets $A \subset \Omega$. An isoperimetric inequality measures the relation between the boundary measure and the measure of a set, by providing a lower bound on $\mathcal{I}(\Omega, d, \mu)$ by some (non-trivial) function $I : [0,1] \to \mathbb{R}_+$. In our manifold-with-density setting, we will always assume that the metric d is given by the induced geodesic distance on (M, g), and write $\mathcal{I} = \mathcal{I}(M, g, \mu)$. When $(\Omega, d) = (\mathbb{R}, |\cdot|)$, we also define $\mathcal{I}^\flat = \mathcal{I}^\flat(\mathbb{R}, |\cdot|, \mu)$ as the pointwise maximal function $\mathcal{I}^\flat : [0,1] \to \mathbb{R}_+ \cup \{+\infty\}$, so that $\mu^+(A) \geqslant \mathcal{I}^\flat(\mu(A))$ for all half lines $A = (-\infty, a)$ and $A = (a, \infty)$ (the difference with the function $\mathcal{I}$ being that the latter is tested on arbitrary Borel sets A).

When $\rho > 0$, sharp isoperimetric inequalities under the $CD(\rho, n+q)$ condition are known and well understood, thanks to the existence of comparison model spaces on which equality is attained. The first such result was obtained by M. Gromov in [12] (reprinted in [13, Appendix C]), who identified the n-sphere as the extremal model space in the constant density case ($q = 0$), thereby extending P. Lévy's isoperimetric inequality on the sphere [15,23]. The case when $q = +\infty$ was treated by Bakry and Ledoux [3] (see also Morgan [20] for a geometric derivation), who showed that the corresponding model space is the real line equipped with a Gaussian density. An extension of these results to $q \in (0, \infty)$ was subsequently obtained by Bayle in [6, Appendix E].

When $\rho \leqslant 0$, the situation is very different, and without requiring some *additional* information on the space (M, g, μ), no isoperimetric inequality can be deduced under the $CD(\rho, n+q)$ condition (in the sense that $\mathcal{I}(M, g, \mu)$ can be arbitrarily small). Various types of information have been considered in the literature, but perhaps the most classical assumption from the view point of Riemannian Geometry is an upper bound on the diameter. By considering domains Ω with bottlenecks, it is immediate to see that again no isoperimetric inequality can be deduced in general, and so requiring that Ω be geodesically convex is a natural assumption; we thus arrive at the $CDD(\rho, n+q, D)$ condition. Various isoperimetric inequalities assuming $CDD(\rho, n+q, D)$ with $\Omega = M$ and $D < \infty$ have been obtain for the classical constant density case $q = 0$ in [8,7,10]. In particular, when $\rho > 0$ and $D < \pi\sqrt{(n-1)/\rho}$, Croke [9] and Bérard, Besson and Gallot [7] obtained improvements over the Gromov–Lévy inequality. Some of these results were extended to $q > 0$ by Bayle in [6].

However, with the exception of the known results under the $CD(\rho, n+q)$ condition when $\rho > 0$, none of the above mentioned results yield *sharp* isoperimetric inequalities for all $v \in (0,1)$. Moreover, most known results fail to capture the behavior of $\mathcal{I}(v)$ for $v \in (0, 1/2]$ close to and away from 0 simultaneously, and miss the optimal inequality by dimension dependent factors. The difficulty when $\rho \leqslant 0$ lies in that there does not seem to be a *good* model space to compare to, as in the Gromov–Lévy or Bakry–Ledoux results. The purpose of this work is to fill this gap, providing a sharp isoperimetric inequality under the $CDD(\rho, n+q, D)$ condition in the entire range $\rho \in \mathbb{R}$, $q \in [0, \infty]$, $D \in (0, \infty]$ and $v \in (0,1)$, in a single unified framework. In particular, for each choice of parameters, we identify the *model spaces* which are extremal for the isoperimetric problem. Our results seem new even in the classical constant-density case ($q = 0$) when $\rho \leqslant 0$ and $D < \infty$ or when $\rho > 0$ and $D < \pi\sqrt{(n-1)/\rho}$.

2. Results

Given $\delta \in \mathbb{R}$, set as usual:

$$s_\delta(t) := \begin{cases} \sin(\sqrt{\delta}t)/\sqrt{\delta}, & \delta > 0, \\ t, & \delta = 0, \\ \sinh(\sqrt{-\delta}t)/\sqrt{-\delta}, & \delta < 0, \end{cases} \qquad c_\delta(t) := \begin{cases} \cos(\sqrt{\delta}t), & \delta > 0, \\ 1, & \delta = 0, \\ \cosh(\sqrt{-\delta}t), & \delta < 0. \end{cases}$$

Given a continuous function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) \geqslant 0$, we denote by $f_+ : \mathbb{R} \to \mathbb{R}_+$ the function coinciding with f between its first non-positive and first positive roots, and vanishing everywhere else.

Definition 3. Given $H, \rho \in \mathbb{R}$ and $m \in (0, \infty]$, set $\delta := \rho/m$ and define

$$J_{H,\rho,m}(t) := \begin{cases} (c_\delta(t) + \frac{H}{m} s_\delta(t))_+^m, & m \in (0,\infty), \\ \exp(Ht - \frac{\rho}{2}t^2), & m = \infty. \end{cases}$$

Remark 1. Observe that since $c_\delta(t) = 1 - \frac{\delta}{2}t^2 + o(\delta)$ and $s_\delta(t) = t + o(\delta)$ as $\delta \to 0$, it follows that $\lim_{m\to\infty} J_{H,\rho,m} = J_{H,\rho,\infty}$. Also observe (with the usual interpretation when $m = \infty$) that $J_{H,\rho,m}$ coincides with the solution J to the following second order ODE, on the maximal interval containing the origin where such a solution exists:

$$-(\log J)'' - \frac{1}{m}\big((\log J)'\big)^2 = -m\frac{(J^{1/m})''}{J^{1/m}} = \rho, \qquad J(0) = 1, \qquad J'(0) = H.$$

Lastly, given a non-negative integrable function f on a closed interval $L \subset \mathbb{R}$, we denote for short $\mathcal{I}(f, L) := \mathcal{I}(\mathbb{R}, |\cdot|, \mu_{f,L})$, where $\mu_{f,L}$ is the probability measure supported in L with density proportional to f there. Similarly, we set $\mathcal{I}^\flat(f, L) := \mathcal{I}^\flat(\mathbb{R}, |\cdot|, \mu_{f,L})$. When $\int_L f(x)\,dx = 0$ we set $\mathcal{I}^\flat(f, L) = \mathcal{I}(f, L) \equiv +\infty$, and when $\int_L f(x)\,dx = +\infty$ we set $\mathcal{I}^\flat(f, L) = \mathcal{I}(f, L) \equiv 0$.

Theorem 2. *Let (M^n, g, μ) satisfy the CDD$(\rho, n+q, D)$ condition with $\rho \in \mathbb{R}$, $q \in [0,\infty]$ and $D \in (0, +\infty]$. Then*

$$\mathcal{I}(M, g, \mu) \geqslant \inf_{H\in\mathbb{R},\, a,b\geqslant 0,\, a+b\leqslant D} \mathcal{I}^\flat\big(J_{H,\rho,n+q-1}, [-a, b]\big), \tag{3}$$

where the infimum is interpreted pointwise on $[0, 1]$. *In fact, the infimum above is always attained* (*when $D = \infty$ at $a = b = \infty$*), *one can always use $b = D - a$, and the $\mathcal{I}^\flat$ may be replaced by $\mathcal{I}$, leading to the same lower bound.*

The bound (3) was deliberately formulated to cover the entire range of values for ρ, n, q and D simultaneously, indicating its universal character, but it may be easily simplified as follows:

Corollary 3. *Under the same assumptions and notation as in Theorem 2, and setting $\delta := \frac{\rho}{n+q-1}$:*

Case 1. $q < \infty$, $\rho > 0$, $D < \pi/\sqrt{\delta}$:

$$\mathcal{I}\big(M^n, g, \mu\big) \geqslant \inf_{\xi\in[0,\pi/\sqrt{\delta}-D]} \mathcal{I}^\flat\big(\sin(\sqrt{\delta}t)^{n+q-1}, [\xi, \xi + D]\big).$$

Case 2. $q < \infty$, $\rho > 0$, $D \geqslant \pi/\sqrt{\delta}$:

$$\mathcal{I}\big(M^n, g, \mu\big) \geqslant \mathcal{I}^\flat\big(\sin(\sqrt{\delta}t)^{n+q-1}, [0, \pi/\sqrt{\delta}\,]\big).$$

Case 3. $q < \infty$, $\rho = 0$, $D < \infty$:

$$\mathcal{I}\big(M^n, g, \mu\big) \geqslant \min\Big(\inf_{\xi\geqslant 0} \mathcal{I}^\flat\big(t^{n+q-1}, [\xi, \xi + D]\big), \mathcal{I}^\flat\big(1, [0, D]\big)\Big).$$

Case 4. $q < \infty$, $\rho < 0$, $D < \infty$:

$$\mathcal{I}\big(M^n, g, \mu\big) \geqslant \min\left\{\begin{array}{l} \inf_{\xi\geqslant 0} \mathcal{I}^\flat(\sinh(\sqrt{-\delta}t)^{n+q-1}, [\xi, \xi + D]), \\ \mathcal{I}^\flat(\exp(\sqrt{-\delta}(n+q-1)t), [0, D]), \\ \inf_{\xi\in\mathbb{R}} \mathcal{I}^\flat(\cosh(\sqrt{-\delta}t)^{n+q-1}, [\xi, \xi + D]) \end{array}\right\}.$$

Case 5. $q = \infty$, $\rho \neq 0$, $D < \infty$:

$$\mathcal{I}\big(M^n, g, \mu\big) \geqslant \inf_{\xi\in\mathbb{R}} \mathcal{I}^\flat\left(\exp\left(-\frac{\rho}{2}t^2\right), [\xi, \xi + D]\right).$$

Case 6. $q = \infty$, $\rho > 0$, $D = \infty$:

$$\mathcal{I}\big(M^n, g, \mu\big) \geqslant \mathcal{I}^\flat\left(\exp\left(-\frac{\rho}{2}t^2\right), \mathbb{R}\right).$$

Case 7. $q = \infty$, $\rho = 0$, $D < \infty$:

$$\mathcal{I}\big(M^n, g, \mu\big) \geqslant \inf_{H\geqslant 0} \mathcal{I}^\flat\big(\exp(Ht), [0, D]\big).$$

In all the remaining cases, we have the trivial bound $\mathcal{I}(M^n, g, \mu) \geqslant 0$.

Note that when q is an integer, $\mathcal{I}^\flat(\sin(\sqrt{\delta}t)^{n+q-1}, [0, \pi/\sqrt{\delta}])$ coincides (by testing spherical caps) with the isoperimetric profile of the $(n+q)$-sphere having Ricci curvature equal to ρ, and so Case 2 with $q = 0$ recovers the Gromov–Lévy isoperimetric inequality [12]; for general $q < \infty$, Case 2 was obtained by Bayle [6, Theorem 3.4.18]. Case 6 recovers the Bakry–Ledoux isoperimetric inequality [3,20]. To the best of our knowledge, all remaining cases are new. To illuminate the transition between Cases 1 and 2, note that if (M^n, g, μ) satisfies the $CD(\rho, n+q)$ condition with $\rho > 0$, the diameter of M is bounded above by $\pi/\sqrt{\delta}$: when $q = 0$ this is the classical Bonnet–Myers theorem (e.g. [11]), which was extended to $q > 0$ by Bakry and Ledoux [4] and Qian [22]. As for the sharpness, we have:

Theorem 4. *For any $n \geqslant 2$, $\rho \in \mathbb{R}$, $q \in [0, \infty]$ and $D \in (0, \infty]$, the lower bounds provided in Corollary 3 (or equivalently, the one provided in Theorem 2) on the isoperimetric profile of (M^n, g, μ) satisfying the $CDD(\rho, n+q, D)$ condition, are sharp, in the sense that they cannot be pointwise improved.*

We conclude that with the exception of Cases 2 and 6 above, there is no *single* model space to compare to, and that a simultaneous comparison to a natural *one parameter family* of model spaces is required, nevertheless yielding a sharp comparison result.

3. Method

It is easy to check that Theorem 4 would hold trivially if the requirement that the bounds are sharp for *any* $n \geqslant 2$ were omitted from its formulation, and if we extend our definitions to include the case of *one-dimensional* manifolds-with-density satisfying the $CDD(\rho, 1+q, D)$ condition, i.e. an open interval $\Omega \subset \mathbb{R}$ of length at most D equipped with a $C^2(\Omega)$ probability density $\Psi > 0$ satisfying:

$$-(\log \Psi)'' - \frac{1}{q}\big((\log \Psi)'\big)^2 = -q\frac{(\Psi^{1/q})''}{\Psi^{1/q}} \geqslant \rho \quad \text{in } \Omega.$$

To prove the sharpness in the higher (topological) dimension case, we emulate the one-dimensional model densities given by Corollary 3 on a geodesically convex domain of an n-dimensional manifold, by thickening arbitrarily slightly in $n-1$ dimensions. When $\rho = 0$ or $q = \infty$ this is very easy to accomplish simply by considering Euclidean space, but for the other cases, we already need to construct a family of rotationally-invariant manifolds endowed with appropriate metrics and densities, and this poses a much greater technical challenge, in part due to the required geodesic convexity of Ω; in fact, the hardest case turns out to be the two-dimensional one.

Our method of proving Theorem 2 is entirely geometric, following the approach set forth by Gromov in [12]. We heavily rely on results from Geometric Measure Theory asserting the regularity of isoperimetric minimizers, both in the interior and on the boundary. To estimate the measure swept out by the normal map emanating from the regular part of the minimizer, we employ a generalized version of the Heintze–Karcher theorem due to V. Bayle [6, Appendix E] and F. Morgan [20].

Applications of these results will be developed in a subsequent work. These include analysis of the asymptotic behaviour of the lower bounds given by Corollary 3 as a function of the parameters $\rho, n+q, D$ and v, and a derivation of corresponding Sobolev inequalities on spaces satisfying the $CDD(\rho, n+q, D)$ condition, improving in many cases the best known bounds.

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