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On solutions of the matrix equations $KX - EXF = BY$ and $MXF^2 + DXF + KX = BY$

Sur les solutions des équations matricielles $KX - EXF = BY$ et $MXF^2 + DXF + KX = BY$

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ABSTRACT

This note studies the solutions of generalized Sylvester equations $KX - EXF = BY$ and $MXF^2 + DXF + KX = BY$, and obtains explicit solutions of the equations by using some matrix transformations and the minimal polynomial of the matrix F.

RÉSUMÉ

Dans cette note on étudie les solutions des équations généralisées de Sylvester $KX - EXF = BY$ et $MFX^2 + DXF + KX = BY$, on donne des expressions explicites des solutions de ces équations en utilisant des transformations matricielles et le polynôme minimal de la matrice F.

1. Introduction

Many control problems, such as pole assignment [2,14,16], and eigenstructure assignment [8,12], can be represented by the following second-order linear systems

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = Bu(t), \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^q$ is the control vector and M, D, K and B are matrices of appropriate dimensions. In certain applications, the matrices M, D and K are called the mass, damping and stiffness matrices, respectively. It can be shown that the linear system (1) is closely related with a second-order Sylvester matrix equation and can be written as

$$MXF^2 + DXF + KX = BY, \tag{2}$$

where $M, D, K \in \mathbf{C}^{n\times n}$, $B \in \mathbf{C}^{n\times q}$ and $F \in \mathbf{C}^{p\times p}$ are known matrices, $X \in \mathbf{C}^{n\times p}$ and $Y \in \mathbf{C}^{q\times p}$ are the matrices to be determined. When $M = 0$ and $D = -E$, the second-order Sylvester matrix equation (2) reduces to the generalized Sylvester matrix equation

$$KX - EXF = BY. \tag{3}$$

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When $M = 0$, $D = -I_n$, $B = I_n$, $Y = W$, the second-order Sylvester matrix equation (2) becomes the normal Sylvester matrix equation

$$KX - XF = W. \tag{4}$$

In addition, by substituting $K = -F^\top$ in (4), the normal Sylvester matrix equation reduces to the well-known Lyapunov matrix equation

$$F^\top X + XF = -W. \tag{5}$$

All the equations mentioned above play an important role in various applied problems. Therefore, despite that numerous algorithms were developed to solve these equations ([3–7,9–11,13,17] etc.), the development of some new algorithms is still of importance. In this note, a simple method for solving Eq. (2) and Eq. (3) is presented by some matrix transformations and the minimal polynomial of the matrix F, and the explicit solutions of the equations are provided.

2. Matrix equation $KX - EXF = BY$

In this section, we discuss the solution of the matrix equation (3). To begin with, we give the following lemma [1]:

Lemma 1. *If $L \in \mathbf{C}^{m\times q}$, $J \in \mathbf{C}^{m\times p}$, then $LZ = J$ has a solution $Z \in \mathbf{C}^{q\times p}$ if and only if $LL^+J = J$. In this case, the general solution of the equation can be described as $Z = L^+J + (I_q - L^+L)U$, where L^+ represents the Moore–Penrose generalized inverse of the matrix L, and $U \in \mathbf{C}^{q\times p}$ is an arbitrary matrix.*

It follows from Lemma 1 that the equation of (3) with unknown matrix Y has a solution if and only if

$$\left(I_n - BB^+\right)KX - \left(I_n - BB^+\right)EXF = 0, \tag{6}$$

when the condition (6) is satisfied, the general solution to the equation of (3) with unknown matrix Y is given by

$$Y = B^+(KX - EXF) + \left(I_q - B^+B\right)T,$$

where $T \in \mathbf{C}^{q\times p}$ is an arbitrary matrix.

Let

$$P_1 = \left(I_n - BB^+\right)K, \qquad Q_1 = \left(I_n - BB^+\right)E,$$

then, the equation of (6) is equivalent to

$$P_1X = Q_1XF. \tag{7}$$

Applying the approach in [15], assume that the columns of the matrix $[G_1, H_1]^\top$ form the basis of the null space of $[Q_1^\top, -P_1^\top]$ (the matrices G_1, H_1 may be found using procedure null.m package MATLAB), then we have

$$G_1Q_1 = H_1P_1. \tag{8}$$

Using the equality (8), we get

$$G_1P_1X = G_1Q_1XF = H_1P_1XF = H_1Q_1XF^2. \tag{9}$$

Let

$$P_2 = G_1P_1, \qquad Q_2 = H_1Q_1, \tag{10}$$

then the equation of (9) becomes

$$P_2X = Q_2XF^2. \tag{11}$$

Similarly, let the columns of the matrix $[G_2, H_2]^\top$ form the basis of the null space of $[Q_2^\top, -P_2^\top]$, that is,

$$G_2Q_2 = H_2P_2. \tag{12}$$

Using the equality (12), we have

$$P_3X = Q_3XF^3,$$

where $P_3 = G_2P_2$, $Q_3 = H_2Q_2$.

A similar procedure can be used to construct the relation with higher degrees of the matrix F,

$$P_k X = Q_k X F^k, \quad k = 1, 2, \ldots, \tag{13}$$

where $P_k = G_{k-1} P_{k-1}$, $Q_k = H_{k-1} Q_{k-1}$, and the columns of the matrix $[G_{k-1}, H_{k-1}]^\top$ form the basis of the null space of $[Q_{k-1}^\top, -P_{k-1}^\top]$, that is,

$$G_{k-1} Q_{k-1} = H_{k-1} P_{k-1}, \quad k = 2, 3, \ldots.$$

It is easily seen that

$$P_k = G_{k-1} G_{k-2} \cdots G_2 G_1 P_1, \qquad Q_k = H_{k-1} H_{k-2} \cdots H_2 H_1 Q_1. \tag{14}$$

Assume that the minimal polynomial of the matrix F is

$$m_F(\lambda) = \lambda^l + f_1 \lambda^{l-1} + \cdots + f_{l-1}\lambda + f_l. \tag{15}$$

Then, by (13), we have

$$\begin{aligned}&(P_l + f_1 H_{l-1} P_{l-1} + f_2 H_{l-1} H_{l-2} P_{l-2} + \cdots + f_{l-1} H_{l-1} H_{l-2} \cdots H_2 H_1 P_1 + f_l Q_l) X \\ &\quad = Q_l X \big(F^l + f_1 F^{l-1} + \cdots + f_{l-1} F + f_l I_p\big) = 0.\end{aligned}$$

In summary of the above discussion and using Lemma 1, we have proved the following result:

Theorem 1. *Let $P_1 = (I_n - BB^+)K$, $Q_1 = (I_n - BB^+)E$. Assume that the matrix $[G_{k-1}, H_{k-1}]$ is of full row rank and satisfies $G_{k-1} Q_{k-1} = H_{k-1} P_{k-1}$, $k = 2, 3, \ldots$, where $P_k = G_{k-1} P_{k-1}$, $Q_k = H_{k-1} Q_{k-1}$, $k = 2, 3, \ldots$. Let the minimal polynomial of the matrix F be given by* (15). *Set $D = P_l + f_1 H_{l-1} P_{l-1} + f_2 H_{l-1} H_{l-2} P_{l-2} + \cdots + f_{l-1} H_{l-1} H_{l-2} \cdots H_2 H_1 P_1 + f_l Q_l$, then the solution of Eq.* (3) *can be expressed as*

$$X = \big(I_n - D^+ D\big) V, \tag{16}$$

$$Y = B^+\big[K\big(I_n - D^+ D\big)V - E\big(I_n - D^+ D\big)VF\big] + \big(I_q - B^+ B\big)T, \tag{17}$$

where $V \in \mathbf{C}^{n\times p}$, $T \in \mathbf{C}^{q\times p}$ are arbitrary matrices.

3. Matrix equation $MXF^2 + DXF + KX = BY$

In this section, we study the solution of the matrix equation (2). Using Lemma 1, the equation of (2) with unknown matrix Y has a solution if and only if

$$\big(I_n - BB^+\big)\big(MXF^2 + DXF + KX\big) = 0, \tag{18}$$

when the condition (18) is satisfied, the general solution to the equation of (2) with unknown matrix Y is given by

$$Y = B^+\big(MXF^2 + DXF + KX\big) + \big(I_q - B^+ B\big)T,$$

where $T \in \mathbf{C}^{q\times p}$ is an arbitrary matrix.

Let

$$\tilde{P}_1 = \begin{bmatrix} -(I_n - BB^+)K & 0 \\ 0 & (I_n - BB^+)M \end{bmatrix}, \qquad \tilde{Q}_1 = \begin{bmatrix} (I_n - BB^+)D & (I_n - BB^+)M \\ (I_n - BB^+)M & 0 \end{bmatrix}, \tag{19}$$

then, the equation of (18) is equivalent to

$$\tilde{P}_1 \begin{bmatrix} X \\ XF \end{bmatrix} = \tilde{Q}_1 \begin{bmatrix} X \\ XF \end{bmatrix} F. \tag{20}$$

By a similar approach in Section 2, we have

$$\tilde{P}_k \begin{bmatrix} X \\ XF \end{bmatrix} = \tilde{Q}_k \begin{bmatrix} X \\ XF \end{bmatrix} F^k, \tag{21}$$

where the matrix $[\tilde{G}_{k-1}, \tilde{H}_{k-1}]$ is of full row rank and is determined alternately by the following relations:

$$\tilde{G}_{k-1} \tilde{Q}_{k-1} = \tilde{H}_{k-1} \tilde{P}_{k-1}, \quad k = 2, 3, \ldots, \tag{22}$$

$$\tilde{P}_k = \tilde{G}_{k-1} \tilde{P}_{k-1}, \qquad \tilde{Q}_k = \tilde{H}_{k-1} \tilde{Q}_{k-1}, \quad k = 2, 3, \ldots. \tag{23}$$

Assume that the minimal polynomial of the matrix F is given by (15). Then, by (21), we have

$$\tilde{D}\begin{bmatrix} X \\ XF \end{bmatrix} = 0, \tag{24}$$

where $\tilde{D} = \tilde{P}_l + f_1\tilde{H}_{l-1}\tilde{P}_{l-1} + f_2\tilde{H}_{l-1}\tilde{H}_{l-2}\tilde{P}_{l-2} + \cdots + f_{l-1}\tilde{H}_{l-1}\tilde{H}_{l-2}\cdots\tilde{H}_2\tilde{H}_1\tilde{P}_1 + f_l\tilde{Q}_l$.

Let

$$\tilde{D} = [P_1, -Q_1].$$

Then the equation of (24) is equivalent to

$$P_1X = Q_1XF, \tag{7}$$

and the solution is given by (16).

By now, we have proved the following result:

Theorem 2. *Let $\tilde{P}_1$, $\tilde{Q}_1$ be given by* (19). *Assume that the matrix $[\tilde{G}_{k-1}, \tilde{H}_{k-1}]$ is of full row rank and satisfies $\tilde{G}_{k-1}\tilde{Q}_{k-1} = \tilde{H}_{k-1}\tilde{P}_{k-1}$, $k = 2, 3, \ldots$, where $\tilde{P}_k = \tilde{G}_{k-1}\tilde{P}_{k-1}$, $\tilde{Q}_k = \tilde{H}_{k-1}\tilde{Q}_{k-1}$, $k = 2, 3, \ldots$. Let the minimal polynomial of the matrix F be given by* (15). *Set $\tilde{D} = \tilde{P}_l + f_1\tilde{H}_{l-1}\tilde{P}_{l-1} + f_2\tilde{H}_{l-1}\tilde{H}_{l-2}\tilde{P}_{l-2} + \cdots + f_{l-1}\tilde{H}_{l-1}\tilde{H}_{l-2}\cdots\tilde{H}_2\tilde{H}_1\tilde{P}_1 + f_l\tilde{Q}_l$ and then partition $\tilde{D}$ as $\tilde{D} = [P_1, -Q_1]$. Then the solution of Eq.* (2) *can be expressed as*

$$X = (I_n - D^+D)V, \tag{25}$$

$$Y = B^+\left[M(I_n - D^+D)VF^2 + D(I_n - D^+D)VF + K(I_n - D^+D)V\right] + (I_q - B^+B)T, \tag{26}$$

where $D = P_l + f_1H_{l-1}P_{l-1} + f_2H_{l-1}H_{l-2}P_{l-2} + \cdots + f_{l-1}H_{l-1}H_{l-2}\cdots H_2H_1P_1 + f_lQ_l$, and $V \in \mathbf{C}^{n\times p}$, $T \in \mathbf{C}^{q\times p}$ are arbitrary matrices.

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