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Complex Analysis

Optimal constant problem in the L^2 extension theorem

Problème de la constante optimale dans le théorème d'extension L^2

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ABSTRACT

In this Note, we solve the optimal constant problem in the L^2-extension theorem with negligible weight on Stein manifolds. As an application, we prove the Suita conjecture on arbitrary open Riemann surfaces.

RÉSUMÉ

Dans cette Note, nous résolvons le problème de la détermination de la constante optimale dans le théorème d'extension L^2 avec poids négligeable sur les variétés de Stein. En application, nous prouvons la conjecture de Suita sur des surfaces de Riemann arbitraires.

1. Introduction and main results

In [16], Ohsawa and Takegoshi proved a famous L^2 extension theorem. Several years later, Ohsawa [14] obtained the L^2 extension theorem with so-called negligible weight ψ, which implies the original Ohsawa–Takegoshi L^2 extension theorem in [16] by taking $\psi = 0$. In the present paper, we solve the optimal constant problem in the L^2 extension theorem on Stein manifolds for arbitrary negligible weight. Our proof is based on the methods of [10] and [22]. The present paper is a continuation of [10] and [22]. Our main result is as follows:

Theorem 1.1. *Let X be a Stein manifold of dimension n. Let $\varphi + \psi$ and ψ be plurisubharmonic functions on X. Assume that w is a holomorphic function on X such that $\sup_X(\psi + 2\log|w|) \leqslant 0$ and $\mathrm{d}w$ does not vanish identically on any branch of $w^{-1}(0)$. Put $H = w^{-1}(0)$ and $H_0 = \{x \in H\colon \mathrm{d}w(x) \neq 0\}$. Then there exists a uniform constant $\mathbf{C} = 1$ such that, for any holomorphic $(n-1)$-form f on H_0 satisfying $c_{n-1}\int_{H_0} e^{-\varphi-\psi} f \wedge \bar{f} < \infty$, where $c_k = (-1)^{\frac{k(k-1)}{2}}(\sqrt{-1})^k$ for $k \in \mathbb{Z}$, there exists a holomorphic n-form F on X satisfying $F = \mathrm{d}w \wedge \tilde{f}$ on H_0 with $\imath^*\tilde{f} = f$ and $c_n\int_X e^{-\varphi} F \wedge \bar{F} \leqslant 2\mathbf{C}\pi c_{n-1}\int_{H_0} e^{-\varphi-\psi} f \wedge \bar{f}$, where $\imath : H_0 \to X$ is the inclusion map.*

Finding the optimal constant $\mathbf{C}$ is one of the interesting issues. In the last two decades, a number of papers have considered this problem, using various methods. For $\psi = 0$, Siu in [18] got an explicit constant $\mathbf{C}$; Manivel in [11], Demailly in [7] and McNeal–Varolin in [13] gave explicit constants on Stein manifolds. In [1], Berndtsson showed that the constant $\mathbf{C}$ of Theorem 1.1 can be taken equal to 4 in the case $\psi = 0$. Somewhat later, Chen in [5] improved the constant $\mathbf{C}$ to 3.3155. In our papers [10] and [22], by developing various techniques used in Berndtsson [1], Demailly [7,9], McNeal–Varolin [13],

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and Siu [18,17] and by reducing the optimal constant problem to an ODE, we obtained $\mathbf{C} \leqslant 1.954$ in Theorem 1.1 for any negligible weight on Stein manifolds. Blocki in [2] recently gave another proof of our theorem in the case of pseudoconvex domains in $\mathbb{C}^n$ with a special negligible weight $\psi = 2G_\Omega(z^n, 0) - 2\log|z^n|$. After that, Blocki in [3] gave an optimal constant $\mathbf{C} = 1$ in Theorem 1.1 on pseudoconvex domains in $\mathbb{C}^n$ with the same negligible weight.

One motivation to estimate the constant $\mathbf{C}$ in Theorem 1.1 comes from the Suita conjecture [20], stated below:

Let Ω be an open Riemann surface which admits a Green function, K_Ω be the Bergman kernel for holomorphic $(1,0)$ forms on Ω and $c_\Omega(z)$ be the logarithmic capacity of Ω with respect to z locally defined by $c_\Omega(z) = \exp\lim_{\xi\to z}(G_\Omega(\xi, z) - \log|\xi - z|)$, where G_Ω is the negative Green function on Ω. Then $(c_\Omega(z))^2|\mathrm{d}z|^2 \leqslant \pi K_\Omega(z)$ for any $z \in \Omega$.

Suita conjecture was motivated to answer a question posed by Sario and Oikawa in their monograph “Capacity functions” published in 1969 about the relation between the Bergman kernel and logarithmic capacity on any open Riemann surface which admits Green function. The relationship between the Suita Conjecture and the extension theorem was observed and explored by Ohsawa [15], and he proved the estimate with $\mathbf{C} = 750$. Using Theorem 1.1, we get

Corollary 1.2. *One has $(c_\Omega(z))^2|\mathrm{d}z|^2 \leqslant \mathbf{C}\pi K_\Omega(z)$ with constant $\mathbf{C} = 1$.*

Recently, for the case of $\Omega \subset\subset \mathbb{C}$, this was proved by Blocki in [3].

2. Some lemmas used in the proof of Theorem 1.1

In this section, we give some lemmas which will be used in the proof of Theorem 1.1.

Lemma 2.1. *Let $(X, \mathrm{d}s_X^2)$ be a Kähler manifold of dimension n with a Kähler metric $\mathrm{d}s_X^2$, $\Omega \Subset X$ be a domain with C^∞ boundary $b\Omega$ and $\Phi \in C^1(\overline{\Omega})$ such that $\partial\bar\partial\Phi$ is a continuous form on $\overline{\Omega}$. Let ρ be a C^∞ defining function for Ω such that $|\mathrm{d}\rho| = 1$ on $b\Omega$. Assume that H is a closed set in X such that the $(2n-1)$-dimensional Hausdorff measure of $H \cap \Omega$ is zero. Let η be a real function in $L^1_{loc}(X) \cap C^0(X \setminus H)$ such that $\partial\bar\partial\eta$ is a current of order zero on X. For any $(n,1)$-form $\alpha = \sum'_{|I|=n} \alpha_{I\bar j}\,\mathrm{d}z^I \wedge \mathrm{d}\bar z^j \in \mathrm{Dom}_\Omega(\bar\partial^*) \cap C^\infty_{(n,1)}(\overline{\Omega})$ one has*

$$\begin{aligned}
&\int_\Omega \eta|\bar\partial_\Phi^*\alpha|^2 e^{-\Phi}\,\mathrm{d}V_X + \int_\Omega \eta|\bar\partial\alpha|^2 e^{-\Phi}\,\mathrm{d}V_X \\
&\quad= \sum_{|J|=1}{}' \sum_{i,j=1}^n \int_\Omega \eta g^{i\bar j}\bar\nabla_j \alpha_{I\bar J}\overline{\bar\nabla_i \alpha^{\bar I J}} e^{-\Phi}\,\mathrm{d}V_X + \sum_{i,j=1}^n \int_{b\Omega} \eta(\partial_i\bar\partial_j\rho)\alpha_I^i\overline{\alpha^{\bar I j}} e^{-\Phi}\,\mathrm{d}S \\
&\qquad+ \sum_{i,j=1}^n \int_\Omega \eta(\partial_i\bar\partial_j\Phi)\alpha_I^i\overline{\alpha^{\bar I j}} e^{-\Phi}\,\mathrm{d}V_X + \sum_{i,j=1}^n \int_\Omega -(\partial_i\bar\partial_j\eta)\alpha_I^i\overline{\alpha^{\bar I j}} e^{-\Phi}\,\mathrm{d}V_X + 2\mathrm{Re}\big(\bar\partial_\Phi^*\alpha, \alpha \llcorner (\bar\partial\eta)^\sharp\big)_{\Omega,\Phi},
\end{aligned} \tag{1}$$

where $(g^{i\bar j})_{n\times n} = \overline{(g_{i\bar j})}^{-1}_{n\times n}$, and $\mathrm{d}V_X$ is the volume form induced from $\mathrm{d}s_X^2$.

We refer to [22] for a proof and more details about the notation. Let T be a current of order 0 on X, α be a differential form on X, Φ be a C^1 function continuous on $\overline{\Omega}$, $\Omega \Subset X$, then the inner product $(T, \alpha)_{\Omega,\Phi}$ is well-defined (see [8]). In the case of a twist factor $\eta \in C^\infty(\overline{\Omega})$ and a weight $\Phi \in C^2(\overline{\Omega})$, the identity (1) was already known to be true (see [18,19,1,12], see also [4,6,21]).

Lemma 2.2. *(See [1,22].) Let $(X, \mathrm{d}s_X^2)$ be a Kähler manifold of dimension n with a Kähler metric $\mathrm{d}s_X^2$, $\Omega \Subset X$ be a strictly pseudoconvex domain in X with C^∞ boundary $b\Omega$ and $\Phi \in C^1(\overline{\Omega})$. Let w be a holomorphic function on X, such that $\mathrm{d}w$ does not vanish identically, and let λ be the current $\bar\partial\frac{1}{w} \wedge \tilde F$, where $\tilde F$ is a holomorphic n-form on X. Assume the inequality $|(\lambda, \alpha)_{\Omega,\Phi}|^2 \leqslant C\int_\Omega |\bar\partial_\Phi^*\alpha|^2 \frac{e^{-\Phi}}{\mu}\,\mathrm{d}V_X$, where $\frac{1}{\mu}$ is an integrable positive function on Ω and C is a constant, holds for all $(n,1)$-form $\alpha \in \mathrm{Dom}_\Omega(\bar\partial^*) \cap \mathrm{Ker}(\bar\partial) \cap C^\infty_{(n,1)}(\overline{\Omega})$. Then there is a solution u to the equation $\bar\partial u = \lambda$ such that $\int_\Omega |u|^2\mu e^{-\Phi}\,\mathrm{d}V_X \leqslant C$.*

3. Proof of Theorem 1.1

Since X is Stein, we can find a sequence of strictly pseudoconvex domains $\{D_v\}_{v=1}^\infty$ with smooth boundaries satisfying $D_v \Subset D_{v+1}$ for all v and $\bigcup_{v=1}^\infty D_v = X$. As $\psi + \log|w|^2 < 0$, we can choose a sequence of numbers $\gamma_v^0 \in (0,1)$ which satisfies $\lim_{v\to\infty}\gamma_v^0 = 1$, such that $(\psi + \gamma_v \log|w|^2)|_{D_v} < 0$ for any $\gamma_v \in [\gamma_v^0, 1)$.

Exactly as in the arguments of [16] and [14], we may assume that $H = H_0$ and that both φ and ψ are smooth on X. Since X is Stein, there is a holomorphic n-form $\tilde F$ on X such that $\tilde F = \mathrm{d}w \wedge \tilde f$ on H with $\iota^*\tilde f = f$. Let $\mathrm{d}s_X^2$ be a Kähler metric on X. The volume form with respect to the induced Kähler metric on H is denoted by $\mathrm{d}V_H$. Let $\eta = s(-\psi - \gamma_v\log|w|^2)$

and $\phi = u(-\psi - \gamma_v \log|w|^2)$, where $s \in C^\infty((0,+\infty))$ satisfies $s \geqslant 0$ and $\lim_{t\to+\infty} s'(t) = 1$, and $u \in C^\infty((0,+\infty))$ satisfies $\lim_{t\to+\infty} u(t) = 0$, $\lim_{t\to+\infty} u'(t) = 0$, $\lim_{t\to+\infty} s(t)u'(t) = 0$, $u''s - s'' > 0$, and $s' - u's \geqslant 0$. Let $\Phi = \varphi + \psi + \phi$.

It is well known that there exists an open neighborhood U of H in X and a holomorphic retraction $r : U \to H$. Since $\mathrm{d}w|_H$ does not vanish, for any point $x \in \overline{D}_v \cap H$, there exists a local coordinate system $(U_x, z_x^1, z_x^2, \ldots, z_x^{n-1}, w)$ of x in X such that $U_x \Subset U$ and $U_x \cap H = \{y \in U_x\colon\ w(y) = 0\}$. Define $V_x = r^{-1}(U_x \cap H) \cap U_x$. There exist $x_1, x_2, \ldots, x_m \in \overline{D}_v \cap H$ such that $\overline{D}_v \cap H \subset \bigcup_{i=1}^m V_{x_i}$. We denote $(V_{x_i}, z_{x_i}^1, z_{x_i}^2, \ldots, z_{x_i}^{n-1}, w)$ simply by $(V_i, z_i^1, z_i^2, \ldots, z_i^{n-1}, w)$. Choose an open set V_{m+1} in X such that $\overline{D}_v \cap H \subset X \setminus \overline{V_{m+1}} \Subset \bigcup_{i=1}^m V_i$. Define $V = X \setminus \overline{V_{m+1}}$. Let $\{\xi_i\}_{i=1}^{m+1}$ be a partition of unity subordinate to the open covering $\{V_i\}_{i=1}^{m+1}$. Then $\operatorname{supp}\xi_i \Subset V_i$ for $i = 1, 2, \ldots, m$ and $\sum_{i=1}^m \xi_i = 1$ on V. Now let $\alpha = \sum'_{|I|=n} \sum_{j=1}^n \alpha_{I\bar{j}}\, \mathrm{d}z^I \wedge \mathrm{d}\bar{z}^j \in \mathrm{Dom}_{D_v}(\bar\partial^*) \cap \mathrm{Ker}(\bar\partial) \cap C^\infty_{(n,1)}(\overline{D_v})$. As $\varphi + \psi$ is a plurisubharmonic function, then by Lemma 2.1 and the Cauchy–Schwarz inequality, we get $\int_{D_v} (\eta + g^{-1})|\bar\partial^*_\Phi \alpha|^2 e^{-\Phi}\, \mathrm{d}V_X \geqslant \sum'_{|I|=n} \sum_{j,k=1}^n \int_{D_v} (-\partial_j\bar\partial_k\eta + \eta\partial_j\bar\partial_k\phi - g(\partial_j\eta)\bar\partial_k\eta)\alpha_I^j \overline{\alpha^{\bar{I}k}} e^{-\Phi}\, \mathrm{d}V_X$, where g is a nonnegative continuous function on D_v, which is positive on $D_v \setminus H$.

We need some calculations to determine g. Let $\partial_n := \frac{\partial}{\partial w}$ and $\partial_k := \frac{\partial}{\partial z^k}$ where $1 \leqslant k \leqslant n-1$. By the Poincaré–Lelong formula, we have $\partial_n\bar\partial_n\eta = -s'(-\psi - \gamma_v\log|w|^2)\gamma_v\pi\delta_{\{w=0\}} - s'(-\psi - \gamma_v\log|w|^2)\partial_n\bar\partial_n\psi + s''(-\psi - \gamma_v\log|w|^2)\partial_n(-\psi - \gamma_v\log|w|^2)\bar\partial_n(-\psi - \gamma_v\log|w|^2)$ and $\partial_j\bar\partial_k\eta = -s'(-\psi - \gamma_v\log|w|^2)\partial_j\bar\partial_k\psi + s''(-\psi - \gamma_v\log|w|^2)\partial_j(-\psi - \gamma_v\log|w|^2)\bar\partial_k(-\psi - \gamma_v\log|w|^2)$, for any $(j,k) \neq (n,n)$, where $\delta_{\{w=0\}}$ is a (n,n) current of order 0, such that $\langle\delta_{\{w=0\}}, \sqrt{-1}\,\mathrm{d}w \wedge \mathrm{d}\bar{w} \wedge v\rangle = 2\int_{\{w=0\}} v$ for any test $(n-1,n-1)$ form v on X. Since $\lim_{t\to+\infty} u'(t) = 0$, we have $\partial_j\bar\partial_k\phi = -u'(-\psi - \gamma_v\log|w|^2)\partial_j\bar\partial_k\psi + u''(-\psi - \gamma_v\log|w|^2)\partial_j(-\psi - \gamma_v\log|w|^2)\bar\partial_k(-\psi - \gamma_v\log|w|^2)$ for any j, k which satisfies $1 \leqslant j, k \leqslant n$. Note that $\alpha_I^n \overline{a^{\bar{I},n}} = |\alpha \llcorner (\mathrm{d}\bar{w})^\sharp|^2$. As ψ is a plurisubharmonic function, $s' - su' \geqslant 0$ and $\lim_{t\to+\infty} s'(t) = 1$, we have $\sum_{1\leqslant j,k\leqslant n}(-\partial_j\bar\partial_k\eta + \eta\partial_j\bar\partial_k\phi - g(\partial_j\eta)\bar\partial_k\eta)\alpha_I^j\overline{\alpha^{\bar{I}k}} \geqslant \gamma_v\pi\delta_{\{w=0\}}|\alpha\llcorner(\mathrm{d}\bar{w})^\sharp|^2 + ((u''s - s'') - gs'^2) \circ (-\psi - \gamma_v\log|w|^2) \sum_{1\leqslant j,k\leqslant n} \partial_j(-\psi - \gamma_v\log|w|^2)\bar\partial_k(-\psi - \gamma_v\log|w|^2)\alpha_I^j\overline{\alpha^{\bar{I}k}}$. Let $g = \frac{u''s - s''}{s'^2} \circ (-\psi - \gamma_v\log|w|)$. We have $\eta + g^{-1} = (s + \frac{s'^2}{u''s - s''}) \circ (-\psi - \gamma_v\log|w|)$, and

$$\int_{D_v} (\eta + g^{-1})\left|\bar\partial^*_\Phi\alpha\right|^2 e^{-\Phi}\, \mathrm{d}V_X \geqslant \int_{D_v} \gamma_v\pi\delta_{\{w=0\}}\left|\alpha\llcorner(\mathrm{d}\bar{w})^\sharp\right|^2 e^{-\Phi}\, \mathrm{d}V_X, \tag{2}$$

where $\eta + g^{-1}$ is assumed to be locally L^1 integrable on $\overline{D}_v$, this will be satisfied for some given s and u.

Let $\lambda = \bar\partial\frac{1}{w} \wedge \tilde{F}$. By inequality (2) and the Cauchy–Schwarz inequality, we can see that $|(\lambda,\alpha)_{D_v,\Phi}|^2 \leqslant |(\pi\delta_{\{w=0\}}\tilde{F}, \alpha\llcorner(\mathrm{d}\bar{w})^\sharp)_{D_v,\Phi}|^2 \leqslant \pi^2\int_{D_v}\delta_{\{w=0\}}\gamma_v^{-1}|\tilde{F}|^2 e^{-\Phi}\, \mathrm{d}V_X \int_{D_v}\gamma_v\delta_{\{w=0\}}|\alpha\llcorner(\mathrm{d}\bar{w})^\sharp|^2 e^{-\Phi}\, \mathrm{d}V_X \leqslant \pi\int_{D_v}\delta_{\{w=0\}}\gamma_v^{-1}|\tilde{F}|^2 e^{-\Phi}\, \mathrm{d}V_X \int_{D_v}(\eta + g^{-1})|\bar\partial^*_\Phi\alpha|^2 e^{-\Phi}\, \mathrm{d}V_X$. Note that $\int_{D_v}\delta_{\{w=0\}}|\tilde{F}|^2 e^{-\Phi}\, \mathrm{d}V_X = 2\int_{D_v\cap H}|f|^2 e^{-\Phi}\, \mathrm{d}V_H$. By Lemma 2.2, we have $(n,0)$-form u_v on D_v satisfying $\bar\partial u_v = \lambda$, such that

$$\int_{D_v} |u_v|^2\left(\eta + g^{-1}\right)^{-1} e^{-\Phi}\, \mathrm{d}V_X \leqslant \pi\int_{D_v}\delta_{\{w=0\}}\gamma_v^{-1}|\tilde{F}|^2 e^{-\Phi}\, \mathrm{d}V_X = \frac{2\pi}{\gamma_v}\int_{D_v\cap H}|f|^2 e^{-\varphi-\psi}\, \mathrm{d}V_H, \tag{3}$$

where $\phi|_{\{w=0\}} \equiv 0$. Let $\mu_1 = e^{\psi + \gamma_v\log|w|^2}$, $\mu = \mu_1 e^\phi$. Assume we can choose η and ϕ such that $\mu \leqslant \mathbf{C}(\eta + g^{-1})^{-1}$, where $\mathbf{C}$ is just the constant in Theorem 1.1. Define $F_v = wu_v$. Since $\bar\partial(u_v - \frac{\tilde{F}}{w}) = 0$, $u_v - \frac{\tilde{F}}{w}$ is a holomorphic n-form. Since $F_v = w(u_v - \frac{\tilde{F}}{w}) + \tilde{F}$, F_v is a holomorphic n-form in D_v satisfying $F_v = \mathrm{d}w \wedge \tilde{f}$ on $D_v \cap H$ with $\iota^*\tilde{f} = f$. We obtain

$$\begin{aligned}\int_{D_v} |F_v|^2 e^{-\varphi}\, \mathrm{d}V_X &= \int_{D_v} |F_v|^2 \frac{\mu_1 e^{-\varphi}}{e^{\psi + \gamma_v\log|w|^2}}\, \mathrm{d}V_X \\ &\leqslant A_v\int_{D_v}|u_v|^2\mu_1 e^\phi e^{-\varphi-\psi-\phi}\, \mathrm{d}V_X = A_v\int_{D_v}|u_v|^2\mu e^{-\varphi-\psi-\phi}\, \mathrm{d}V_X,\end{aligned} \tag{4}$$

where A_v are positive numbers which satisfy $\lim_{v\to\infty} A_v = 1$ when we choose γ_v near enough to 1. By inequalities (3) and (4), we obtain that $\int_{D_v}|F_v|^2 e^{-\varphi}\, \mathrm{d}V_X \leqslant \frac{2\mathbf{C}A_v\pi}{\gamma_v}\int_{D_v\cap H}|f|^2 e^{-\varphi-\psi}\, \mathrm{d}V_H$, under the assumption $\mu \leqslant \mathbf{C}(\eta + g^{-1})^{-1}$ where $\phi|_{\{w=0\}} \equiv 0$. It suffices to find η and ϕ such that $(\eta + g^{-1}) \leqslant \mathbf{C}e^{-\psi-\gamma_v\log|w|^2}e^{-\phi} = \mathbf{C}\mu^{-1}$ on X. Let $t = -\psi - \gamma_v\log|w|^2$. As $\eta = s(-\psi - \gamma_v\log|w|^2)$ and $\phi = u(-\psi - \gamma_v\log|w|^2)$, we have $(\eta + g^{-1})e^{\psi + \gamma_v\log|w|^2}e^\phi = (s + \frac{s'^2}{u''s - s''})e^{-t}e^u \circ (-\psi - \gamma_v\log|w|^2)$. We get an ODE:

$$\left(s + \frac{s'^2}{u''s - s''}\right)e^{u-t} = \mathbf{C}, \tag{5}$$

where $t \in [0,+\infty)$, and $\mathbf{C} = 1$. When $u = 0$, the equation is exactly the ODE that we considered in our articles [10] and [22].

Note that $s' - su' = e^u(se^{-u})'$ and $u''s - s'' + s'u' = -(e^u(se^{-u})')'$, one can simply solve the ODE (5) to get $u = -\log(1 - e^{-t})$ and $s = \frac{t}{1 - e^{-t}} - 1$. Note that ϕ is in $C^1(\overline{D}_v)$ and $\partial\bar\partial\phi$ is continuous on $\overline{D}_v$. Then Φ is in $C^1(\overline{D}_v)$ and

$\partial\bar{\partial}\Phi$ is continuous on $\overline{D}_v$. We can also see that $\partial\bar{\partial}\eta$ is of order 0. One can check that $s' - su' \geqslant 0$ and $\lim_{t\to+\infty} s(t)u'(t) = 0$ holds, and $\eta + g^{-1}$ is locally L^1 integrable on $\overline{D}_v$.

Now we have proved that there exists holomorphic n-form F_v on D_v satisfying $F_v = \mathrm{d}w \wedge \tilde{f}$ on $D_v \cap H$ with $\iota^*\tilde{f} = f$ and $\int_{D_v} |F_v|^2 e^{-\varphi}\,\mathrm{d}V_X \leqslant \frac{2A_v\mathbf{C}\pi}{\gamma_v} \int_{D_v\cap H} |f|^2 e^{-\varphi-\psi}\,\mathrm{d}V_H \leqslant \frac{2A_v\mathbf{C}\pi}{\gamma_v} \int_H |f|^2 e^{-\varphi-\psi}\,\mathrm{d}V_H$. Define $F_v = 0$ on $X\backslash D_v$. As $\lim_{v\to\infty}\gamma_v = \lim_{v\to\infty} A_v = 1$, then the weak limit of some weakly convergent subsequence of $\{F_v\}_{v=1}^{\infty}$ gives us a holomorphic n-form F on X satisfying $F = \mathrm{d}w \wedge \tilde{f}$ on H with $\iota^*\tilde{f} = f$ and $c_n \int_X e^{-\varphi} F \wedge \bar{F} = \int_X |F|^2 e^{-\varphi}\,\mathrm{d}V_X \leqslant 2\mathbf{C}\pi \int_H |f|^2 e^{-\varphi-\psi}\,\mathrm{d}V_H = 2\mathbf{C}\pi c_{n-1} \int_H e^{-\varphi-\psi} f \wedge \bar{f}$. In conclusion, we have proved Theorem 1.1 with constant $\mathbf{C} = 1$.

4. Proof of Corollary 1.2

It is well known that the Bergman kernel K_Y on a complex manifold Y can be defined by holomorphic $(n, 0)$ forms, i.e., $K_Y := \sum_i e_i \otimes \bar{e}_i$, where $\{e_i\}_{i=1,2,\dots}$ are holomorphic $(n, 0)$ forms on Y satisfying $c_n \int_Y \frac{e_i}{\sqrt{2^n}} \wedge \frac{\bar{e}_j}{\sqrt{2^n}} = \delta_i^j$. First we prove a special case of the corollary, and then, we prove the general case.

For the special case, we assume that for any given point $z_0 \in \Omega$, there exists a holomorphic function w on Ω, which satisfies $w(z_0) = 0$, $\mathrm{d}w|_{z_0} \neq 0$, and $w|_{\Omega\backslash z_0} \neq 0$. Take w as the local coordinate on a neighborhood of z_0. Then the Suita conjecture becomes $(c_\Omega(z_0))^2 |\mathrm{d}w|^2 \leqslant \pi K_\Omega(z_0)$. Let $X = \Omega$, $\varphi = 0$ and $\psi(z) = 2G_\Omega(z, z_0) - 2\log|w|$ in Theorem 1.1. This theorem tells us that there exists a holomorphic $(1, 0)$ form F on Ω, such that $F|_{z_0} = \mathrm{d}w$ and $\int_\Omega \sqrt{-1} F \wedge \bar{F} \leqslant \frac{2\mathbf{C}\pi}{(c_\Omega(z_0))^2}$. Since $K_\Omega(z_0) \geqslant \frac{2 f \otimes \bar{f}|_{z_0}}{\sqrt{-1}\int_\Omega f \wedge \bar{f}}$ for any nonzero holomorphic $(1, 0)$ form f on Ω, Corollary 1.2 is proved in that case.

Now we prove the general case. Note that Ω is Stein. It is then known that for any given point $z_0 \in \Omega$, one can choose a holomorphic function w on Ω such that $w(z_0) = 0$ and $\mathrm{d}w|_{z_0} \neq 0$. As $\{w = 0\}$ is discrete in Ω, the capacities of the Riemann surface $\Omega' := \Omega \setminus (\{w = 0\} \setminus z_0)$ and Ω are the same, and so are their Bergman kernels. The special case already proved applied to $\Omega' \ni z_0$ implies Corollary 1.2.

References

[1] B. Berndtsson, The extension theorem of Ohsawa–Takegoshi and the theorem of Donnelly–Fefferman, Ann. L'Inst. Fourier (Grenoble) 46 (4) (1996) 1083–1094.
[2] Z. Blocki, On the Ohsawa–Takegoshi extension theorem, preprint, 2012.
[3] Z. Blocki, Suita conjecture and the Ohsawa–Takegoshi extension theorem, preprint, 2012.
[4] J. Cao, M.-C. Shaw, L. Wang, Estimates for the $\bar{\partial}$-Neumann problem and nonexistence of C^2 Levi-flat hypersurfaces in CP^n, Math. Z. 248 (2004) 183–221.
[5] B. Chen, A remark on an extension theorem of Ohsawa, Chin. Ann. Math., Ser. A 24 (2003) 129–134 (in Chinese).
[6] S.-C. Chen, M.-C. Shaw, Partial Differential Equations in Several Complex Variables, AMS/IP, 2001.
[7] J.-P. Demailly, On the Ohsawa–Takegoshi–Manivel L^2 extension theorem, in: Proceedings of the Conference in Honour of the 85th Birthday of Pierre Lelong, September 1997, Paris, in: Progress in Mathematics, Birkhäuser, 2000.
[8] J.-P. Demailly, Complex analytic and differential geometry, http://www-fourier.ujf-grenoble.fr/~demailly/books.html.
[9] J.-P. Demailly, Analytic Methods in Algebraic Geometry, Higher Education Press, Beijing, 2010.
[10] Q.A. Guan, X.Y. Zhou, L.F. Zhu, On the Ohsawa–Takegoshi L^2 extension theorem and the twisted Bochner–Kodaira identity, C. R. Acad. Sci. Paris, Ser. I 349 (2011) 797–800.
[11] L. Manivel, Un théorème de prolongement L^2 de sections holomorphes d'un fibré vectoriel, Math. Zeitschrift 212 (1993) 107–122.
[12] J. McNeal, On large values of L^2 holomorphic functions, Math. Res. Lett. 3 (2) (1996) 247–259.
[13] J. McNeal, D. Varolin, Analytic inversion of adjunction: L^2 extension theorems with gain, Ann. L'Inst. Fourier (Grenoble) 57 (3) (2007) 703–718.
[14] T. Ohsawa, On the extension of L^2 holomorphic functions. III. Negligible weights, Math. Z. 219 (2) (1995) 215–225.
[15] T. Ohsawa, Addendum to "On the Bergman kernel of hyperconvex domains", Nagoya Math. J. 137 (1995) 145–148.
[16] T. Ohsawa, K. Takegoshi, On the extension of L^2 holomorphic functions, Math. Z. 195 (1987) 197–204.
[17] Y.-T. Siu, Complex-analyticity of harmonic maps, vanishing and Lefschetz theorems, J. Differential Geometry 17 (1982) 55–138.
[18] Y.-T. Siu, The Fujita conjecture and the extension theorem of Ohsawa–Takegoshi, in: Geometric Complex Analysis, Hayama, World Scientific, 1996, pp. 577–592.
[19] Y.-T. Siu, Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semipositively twisted plurigenera for manifolds not necessarily of general type, in: Complex Geometry, Göttingen, 2000, Springer, Berlin, 2002, pp. 223–277.
[20] N. Suita, Capacities and kernels on Riemann surfaces, Arch. Ration. Mech. Anal. 46 (1972) 212–217.
[21] E. Straube, Lectures on the L^2-Sobolev Theory of the $\bar{\partial}$-Neumann Problem, ESI Lectures in Mathematics and Physics, European Mathematical Society, Zürich, 2010.
[22] L.F. Zhu, Q.A. Guan, X.Y. Zhou, On the Ohsawa–Takegoshi L^2 extension theorem and the twisted Bochner–Kodaira identity with a non-smooth twist factor, J. Math. Pures Appl. 97 (6) (2012) 579–601.