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Number Theory

On the sum of distinct primes or squares of primes ☆

Sur les sommes de premiers et de carrés de premiers distincts

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ABSTRACT

In 1965 Erdős introduced $f_2(s)$: $f_2(s)$ is the smallest integer such that every $l > f_2(s)$ is the sum of s distinct primes or squares of primes where a prime and its square are not both used. We prove that for all sufficiently large s, $f_2(s) \leqslant p_2 + p_3 + \cdots + p_{s+1} + 3106$, and the set of s with the equality has the density 1.

RÉSUMÉ

En 1965 Paul Erdős a introduit la valeur $f_2(s)$ comme le plus petit entier tel que tout entier $l > f_2(s)$ est la somme de s premiers ou carrés de premiers distincts, où un nombre premier et son carré ne sont simultanément utilisés. Nous démontrons que pour tout s suffisamment grand on a $f_2(s) \leqslant p_2 + p_3 + \cdots + p_{s+1} + 3106$ et que l'ensemble des s réalisant l'égalité est de densité 1.

1. Introduction

Denote by μ_s the least integer such that every integer $\ell > \mu_s$ is the sum of exactly s integers > 1 which are pairwise relatively prime. In 1964, Sierpiński [4] asked for determination of μ_s. In order to study Sierpiński's problem, P. Erdős [3] introduced $f_1(s)$, $f_2(s)$ and $f_3(s)$. Denote by $f_1(s)$ the least integer such that every integer $\ell > f_1(s)$ is the sum of s distinct primes; $f_2(s)$ is the smallest integer such that every $\ell > f_2(s)$ is the sum of s distinct primes or squares of primes where a prime and its square are not both used; and $f_3(s)$ is the least integer such that every integer $\ell > f_3(s)$ is the sum of s distinct integers > 1 which are pairwise relatively prime. Let $p_1 = 2$, $p_2 = 3, \ldots$ be the sequence of consecutive primes. Clearly, $\mu_s = f_3(s) \leqslant f_2(s) \leqslant f_1(s)$. We have determined μ_s for all s.

In 1965, P. Erdős [3] proved that $f_2(s) < p_2 + p_3 + \cdots + p_{s+1} + C$, where C is an absolute constant.

In this Note, the following results are proved:

Theorem 1. (a) *For* $s \geqslant s_0$, $f_2(s) \leqslant p_2 + p_3 + \cdots + p_{s+1} + 3106$;
(b) *If* $s \geqslant s_0$ *and* $p_{s+2} - p_{s+1} > 3106$, *then* $f_2(s) = p_2 + p_3 + \cdots + p_{s+1} + 3106$;
(c) *The set of* s *with* $f_2(s) = p_2 + p_3 + \cdots + p_{s+1} + 3106$ *has the density* 1.

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2. Proofs

Lemma 1. *(See [2].) For $x \geqslant 24$ there exists a prime in $(x, \sqrt{\frac{3}{2}}x]$.*

Lemma 2. *Every even number $n \geqslant 3106$ can be expressed as the sum of distinct $p_k^2 - p_k$ $(k \geqslant 2)$. The number 3106 cannot be expressed as the sum of distinct $p_k^2 - p_k$ $(k \geqslant 2)$.*

Proof. The proof is by induction on the even number n. Let

$$V_1 = \{0\}, \qquad V_{i+1} = V_i \cup \left(V_i + p_{i+1}^2 - p_{i+1}\right), \quad i = 1, 2, \ldots.$$

By Mathematica, we find that $[3108, 10\,000] \cap (2\mathbb{Z}) \subseteq V_{30}$ and $3106 \notin V_{30}$. Hence, if n is even with $3108 \leqslant n \leqslant 10\,000$, then n can be expressed as the sum of distinct $p_k^2 - p_k$ $(k \geqslant 2)$.

Now assume that for any even integer n with $3108 \leqslant n < 2m$ $(2m > 10\,000)$, n can be expressed as the sum of distinct $p_k^2 - p_k$ $(k \geqslant 2)$.

Since $2m - 3108 > 10\,000 - 3108 > 83^2 - 83$, there exists a prime $p_u \geqslant 83$ with

$$p_u^2 - p_u \leqslant 2m - 3108 < p_{u+1}^2 - p_{u+1}. \tag{1}$$

Then

$$3108 \leqslant 2m - \left(p_u^2 - p_u\right) < 2m.$$

By the induction hypothesis, we have

$$2m - \left(p_u^2 - p_u\right) = \sum_{i=1}^{t} \left(q_i^2 - q_i\right),$$

where $q_1 < \cdots < q_t$ are distinct odd primes. Hence

$$2m = \sum_{i=1}^{t} \left(q_i^2 - q_i\right) + \left(p_u^2 - p_u\right). \tag{2}$$

By (1) and Lemma 1, we have

$$2m < p_{u+1}^2 - p_{u+1} + 3108 \leqslant \frac{3}{2} p_u^2 - p_u + 3108 < 2\left(p_u^2 - p_u\right).$$

So $q_t^2 - q_t < p_u^2 - p_u$. Thus, by (2), $2m$ can be expressed as the sum of distinct $p_k^2 - p_k$ $(k \geqslant 2)$.

If 3106 can be expressed as the sum of distinct $p_k^2 - p_k$ $(k \geqslant 2)$, then $p_k^2 - p_k \leqslant 3106$. Then $k < 30$ and $3106 \in V_{30}$, a contradiction. This completes the proof of Lemma 2. □

Lemma 3. *(See [3].) We have $f_2(s) < p_2 + p_3 + \cdots + p_{s+1} + C$, where C is an absolute constant.*

Proof of Theorem 1. (a) Let s_0 be the least integer with $50 p_{s_0} > C$, where C is as in Lemma 3. We may assume that $C > 3106$. Then $s_0 \geqslant 16$ and $p_{s_0} \geqslant 53$. Let $s \geqslant s_0$ and ℓ be an integer with

$$p_2 + p_3 + \cdots + p_{s+1} + 3106 < \ell \leqslant p_2 + p_3 + \cdots + p_{s+1} + C.$$

Write

$$\ell = p_2 + p_3 + \cdots + p_{s+1} + n.$$

Then $3106 < n \leqslant C$ and $p_s^2 - p_s \geqslant 52 p_s > p_{s+1} + C > p_{s+1} + n - 2 > n$.

Case 1: n is even. By Lemma 2, we have

$$n = \sum_{k=2}^{\infty} \left(p_k^{\alpha_k} - p_k\right), \quad \alpha_k \in \{1, 2\}.$$

Since $p_s^2 - p_s > n$, we have $\alpha_k = 1$ for all $k \geqslant s$. Thus

$$\ell = p_2 + p_3 + \cdots + p_{s+1} + n = \sum_{k=2}^{s+1} p_k^{\alpha_k}, \quad \alpha_k \in \{1, 2\}.$$

Case 2: n is odd. By Lemma 2, we have

$$p_{s+1} + n - 2 = \sum_{k=2}^{\infty} \left(p_k^{\alpha_k} - p_k\right), \quad \alpha_k \in \{1, 2\}.$$

Since $p_s^2 - p_s > p_{s+1} + n - 2$, we have $\alpha_k = 1$ for all $k \geqslant s$. Thus

$$\ell = p_2 + p_3 + \cdots + p_{s+1} + n = p_1 + \sum_{k=2}^{s} p_k^{\alpha_k}, \quad \alpha_k \in \{1, 2\}.$$

By Cases 1 and 2, for all $s \geqslant s_0$, we have $f_2(s) \leqslant p_2 + p_3 + \cdots + p_{s+1} + 3106$.

(b) Assume that $p_{s+2} - p_{s+1} > 3106$. Suppose that

$$p_2 + p_3 + \cdots + p_{s+1} + 3106 = q_1^{\alpha_1} + \cdots + q_s^{\alpha_s}$$

with all $\alpha_k \in \{1, 2\}$ and $q_1, \ldots, q_s$ are distinct primes. By comparing the parities of both sides, we have that $q_1, \ldots, q_s$ are distinct odd primes. Thus $q_i \geqslant p_{i+1}$ $(1 \leqslant i \leqslant s)$. If $q_s > p_{s+1}$, then

$$3106 = \sum_{i=1}^{s} \left(q_i^{\alpha_i} - p_{i+1}\right) \geqslant q_s^{\alpha_s} - p_{s+1} \geqslant p_{s+2} - p_{s+1} > 3106,$$

a contradiction. Hence $q_s \leqslant p_{s+1}$. Thus $q_i = p_{i+1}$ for all $1 \leqslant i \leqslant s$. So

$$3106 = \sum_{k=2}^{s+1} \left(p_k^{\alpha_k} - p_k\right),$$

this contradicts Lemma 2. Therefore $p_2 + p_3 + \cdots + p_{s+1} + 3106$ is not the sum of s distinct primes or squares of primes where a prime and its square are not both used. So, by (a), we have $f_2(s) = p_2 + p_3 + \cdots + p_{s+1} + 3106$.

(c) It follows from the fact that the number of primes $p \leqslant x$ with $p + k$ being prime is $O(x/(\log x)^2)$ for each $k = 2, 4, 6, \ldots, 3106$ (see [1]).

This completes the proof of Theorem 1. □

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