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The explicit equivalence between the standard and the logarithmic star product for Lie algebras, I

Une équivalence explicite entre les produit-étoilés standard et logarithmique pour une algèbre de Lie, I

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ABSTRACT

The purpose of this note is to establish an explicit equivalence between two star products $\star$ and $\star_{\log}$ on the symmetric algebra $\mathrm{S}(\mathfrak{g})$ of a finite-dimensional Lie algebra $\mathfrak{g}$ over a field $\mathbb{K} \supset \mathbb{C}$ associated with the standard angular propagator and the logarithmic one respectively: the differential operator of infinite order with constant coefficients realizing the equivalence is related to the incarnation of the Grothendieck–Teichmüller group considered by Kontsevich (1999) in [5, Theorem 7]. We present in the first part the main result, and devote the second part to its proof.

RÉSUMÉ

Dans cette note, on construit explicitement une équivalence entre les deux produits-étoilés $\star$ et $\star_{\log}$ sur l'algèbre symétrique $\mathrm{S}(\mathfrak{g})$ associée à une algèbre de Lie $\mathfrak{g}$ de dimension finie sur un corps $\mathbb{K} \supset \mathbb{C}$, construits en utilisant le propagateur angulaire standard et le propagateur logarithmique respectivement : l'operateur differentiel d'ordre infini à coéfficients constants réalisant cette équivalence est relié à l'incarnation du groupe de Grothendieck–Teichmüller considérée par Kontsevich (1999) dans [5, Theorem 7]. On présente dans cette première partie le résultat principal, dont la démonstration sera donnée dans la deuxième partie.

1. Introduction

For a general finite-dimensional Lie algebra $\mathfrak{g}$ over a field $\mathbb{K} \supset \mathbb{C}$, we consider its symmetric algebra $A = \mathrm{S}(\mathfrak{g})$.

Deformation quantization *à la* Kontsevich [6] permits to endow A with an associative, non-commutative product $\star$, and there is an isomorphism of associative algebras $\mathcal{I}$ from $(A, \star)$ to $(\mathrm{U}(\mathfrak{g}), \cdot)$. The algebra isomorphism $\mathcal{I}$ by [6, Section 8.3] and [8] is the composition of the Poincaré–Birkhoff–Witt (PBW for short) isomorphism (of vector spaces) with the well-known Duflo element $\sqrt{j(\bullet)}$ in the completed symmetric algebra $\widehat{\mathrm{S}}(\mathfrak{g}^*)$.

In this short note, which takes inspiration from recent (unpublished) results [1,2] on the singular logarithmic propagator proposed by Kontsevich in [5, Section 4.1, F)], we discuss the relationship between the star products $\star$ and $\star_{\log}$ on A, where $\star_{\log}$ is the star product associated with the logarithmic propagator.

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The two star products $\star$ and $\star_{\log}$ on A are equivalent because both $(A,\star)$ and $(A,\star_{\log})$ are isomorphic to $(\mathrm{U}(\mathfrak{g}),\cdot)$. We produce here the explicit form of the aforementioned equivalence *via* a translation-invariant, invertible differential operator of infinite order on A depending on the odd traces of the adjoint representation of $\mathfrak{g}$.

The main result is a consequence of the logarithmic version of the formality result in presence of two branes from [4] and of the application discussed in [3] ("Deformation quantization with generators and relations"). Here a *caveat* is necessary: we do not prove here the general logarithmic formality in presence of two branes, but only discuss its main features in the present framework and provide explicit formulæ with a sketch of the main technicalities.

The present result provides a different insight to the incarnation of the Grothendieck–Teichmüller group in deformation quantization considered in [5, Theorem 7]. Observe that, quite differently from [5], here odd traces of the adjoint representation of $\mathfrak{g}$ appear non-trivially, because we are not dealing with the Chevalley–Eilenberg cohomology of $\mathfrak{g}$ with values in A.

2. Notation and conventions

For a field $\mathbb{K}\supset\mathbb{C}$, we denote by $\mathfrak{g}$ a finite-dimensional Lie algebra over $\mathbb{K}$ of dimension d; by $\{x_i\}$ we denote a $\mathbb{K}$-basis of $\mathfrak{g}$. With $\mathfrak{g}$ we associate the (linear) Poisson variety $X=\mathfrak{g}^*$ over $\mathbb{K}$ endowed with the Kirillov–Kostant Poisson bivector field π. We denote by $\mathrm{ad}(\bullet)$ the adjoint representation of $\mathfrak{g}$ on itself; further, for $n\geqslant 1$, we set $c_n(x)=\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(\bullet)^n)$, and c_n belongs to $\mathrm{S}(\mathfrak{g}^*)$. Finally, $\zeta(\bullet)$ and $\Gamma(\bullet)$ denote the Riemann ζ-function and the Γ-function, respectively.

3. An equivalence of star products in the Lie algebra case

For $\mathfrak{g}$ as in Section 2, we consider the Poisson algebra $A=\mathbb{K}[X]=\mathrm{S}(\mathfrak{g})$ endowed with the linear Kirillov–Kostant Poisson bivector field π. We first quickly recall the construction of the star products $\star$ and $\star_{\log}$; later on, we construct an explicit algebra isomorphism from $(A,\star)$ and $(A,\star_{\log})$ to $(\mathrm{U}(\mathfrak{g}),\cdot)$, focusing in particular on $(A,\star_{\log})$.

3.1. Explicit formulæ for the products $\star$ and $\star_{\log}$

Let $X=\mathbb{K}^d$ and $\{x_i\}$ a system of global coordinates on X, for $\mathbb{K}$ as in Section 2.

For a pair (n,m) of non-negative integers, by $\mathcal{G}_{n,m}$ we denote the set of admissible graphs of type (n,m), see [6, Section 6.1] for more details. By $E(\Gamma)$ we denote the set of edges of Γ in $\mathcal{G}_{n,m}$.

We denote by $C^+_{n,m}$, resp. $\bar{C}^+_{n,m}$, the configuration space of n points in the complex upper half-plane $\mathbb{H}^+$ and m ordered points on the real axis $\mathbb{R}$ *modulo* the componentwise action of rescalings and real translations, resp. its compactification *à la* Fulton–MacPherson, see [6, Section 5] for a detailed exposition. For $2n+m-2\geqslant 0$, $\bar{C}^+_{n,m}$ is a compact, oriented, smooth manifold with corners of dimension $2n+m-2$.

We denote by ω, resp. $\omega_{\log}$ the closed, real-valued 1-form

$$\omega(z_1,z_2)=\frac{1}{2\pi}\,\mathrm{d}\arg\left(\frac{z_1-z_2}{\bar{z}_1-z_2}\right),\quad \text{resp. }\omega_{\log}(z_1,z_2)=\frac{1}{2\pi i}\,\mathrm{d}\log\left(\frac{z_1-z_2}{\bar{z}_1-z_2}\right),\quad (z_1,z_2)\in\left(\mathbb{H}^+\sqcup\mathbb{R}\right)^2,\ z_1\neq z_2,$$

where $\arg(\bullet)$ denotes the $[0,2\pi)$-valued argument function on $\mathbb{C}\smallsetminus\{0\}$ such that $\arg(i)=\pi/2$, and $\log(\bullet)$ denotes the corresponding logarithm function, such that $\log(z)=\ln(|z|)+i\arg(z)$.

The 1-form ω extends to a smooth, closed 1-form on $\bar{C}^+_{2,0}$, such that (i) when the two arguments approach to each other in $\mathbb{H}^+$, ω equals the normalized volume form $\mathrm{d}\varphi$ on S^1 and (ii) when the first argument approaches $\mathbb{R}$, ω vanishes.

On the other hand, $\omega_{\log}$ extends smoothly to all boundary strata of $\bar{C}^+_{2,0}$ (*e.g.* through a direct computation, one sees that $\omega_{\log}$ vanishes, when its first argument approaches $\mathbb{R}$ and coincides with ω when the second argument approaches $\mathbb{R}$) except the one corresponding to the collapse of its two arguments in $\mathbb{H}^+$, where it has a complex pole of order 1.

The standard propagator ω has been introduced and discussed in [6, Section 6.2]; the logarithmic propagator $\omega_{\log}$ has been first introduced in [5, Section 4.1, F)].

We introduce $T_{\mathrm{poly}}(X)=A[\theta_1,\dots,\theta_d]$, $A=C^\infty(X)$, for a set $\{\theta_i\}$ of graded variables of degree 1 commuting with A and anticommuting among themselves. We further consider the well-defined linear endomorphism τ of $T_{\mathrm{poly}}(X)^{\otimes 2}$ of degree -1 defined *via* $\tau=\partial_{\theta_i}\otimes\partial_{x_i}$.

With Γ in $\mathcal{G}_{n,m}$ such that $|E(\Gamma)|=2n+m-2$, γ_i, $i=1,\dots,n$, elements of $T_{\mathrm{poly}}(X)$ and a_j, $j=1,\dots,m$, elements of A, we associate two maps $\mathcal{U}_\Gamma$, $\mathcal{U}^{\log}_\Gamma$ *via*

$$\left(\mathcal{U}^{(\log)}_\Gamma(\gamma_1,\dots,\gamma_n)\right)(a_1\otimes\cdots\otimes a_m)=\mu_{m+n}\left(\int_{C^+_{n,m}}\omega^{(\log)}_{\tau,\Gamma}(\gamma_1\otimes\cdots\otimes\gamma_n\otimes a_1\otimes\cdots\otimes a_m)\right),$$

$$\omega^{(\log)}_{\tau,\Gamma}=\prod_{e\in E(\Gamma)}\pi_e^*\left(\omega^{(\log)}\right)\otimes\tau_e.$$

Here, τ_e is the graded endomorphism of $T_{\mathrm{poly}}(X)^{\otimes(m+n)}$ acting as τ on the two factors of $T_{\mathrm{poly}}(X)$ corresponding to the initial and final point of the edge e, π_e is the projection from $C^+_{n,m}$ onto $C^+_{2,0}$ corresponding to e, and μ_{m+n} is the multiplication map from $T_{\mathrm{poly}}(X)^{m+n}$ to $T_{\mathrm{poly}}(X)$, followed by the projection from $T_{\mathrm{poly}}(X)$ onto A.

Theorem 3.1. *For a Poisson bivector field π on X and a formal parameter $\hbar$, the formula*

$$f_1 \star_{\hbar,(\log)} f_2 = \sum_{n\geqslant 0} \frac{\hbar^n}{n!} \sum_{\Gamma\in\mathcal{G}_{n,2}} \big(\mathcal{U}_\Gamma^{(\log)}(\underbrace{\pi,\dots,\pi}_{n})\big)(f_1,f_2), \quad f_i \in A,\ i=1,2, \tag{1}$$

defines a $\mathbb{K}[\![\hbar]\!]$-linear, associative product on $A_\hbar = A[\![\hbar]\!]$.

We refer to [6, Sections 1, 2] for a detailed discussion of (1) for the standard propagator. The following Digression A contains a sketch of the technical arguments for the construction and properties of (1) in the logarithmic case, explained in more detail in [1,2].

Digression A. On the logarithmic propagator(s)

Let us review the main results of [1,2] for the convenience of the reader by pointing out the main technical details.

Convergence of the integral weights $\varpi_\Gamma^{\log}$, for Γ admissible of type (n,m) and $|E(\Gamma)| = 2n+m-2$ in the logarithmic case follows from the fact that the integrand $\omega_\Gamma^{\log}$ on $C^+_{n,m}$ extends to a complex-valued, real analytic form of top degree on the compactified configuration space $\bar{C}^+_{n,m}$.

We must prove that $\omega_\Gamma^{\log}$ extends to all boundary strata of $\bar{C}^+_{n,m}$: because of the boundary properties of $\omega_{\log}$ (*i.e.* $\omega_{\log}$ has a pole of order 1 along the stratum corresponding to the collapse of its two arguments inside $\mathbb{H}^+$), the main technical point concerns the extension to boundary strata describing the collapse of clusters of at least two points in $\mathbb{H}^+$ at different "speeds" to single points in $\mathbb{H}^+$.

By introducing polar coordinates (ρ_i,φ_i), $i=1,\dots,k$, for each cluster of collapsing points near such a boundary stratum, the possible poles in $\omega_\Gamma^{\log}$ take the form

$$\frac{1}{2\pi i}\frac{\mathrm{d}\rho_i}{\rho_i} + \frac{\mathrm{d}\varphi_i}{2\pi} + \cdots, \quad i=1,\dots,k,$$

where $\cdots$ denotes a complex-valued, real analytic 1-form. The angle differential $\mathrm{d}\varphi_i$ appears without a factor ρ_i only when paired to the corresponding singular logarithmic differential $\mathrm{d}\rho_i/\rho_i$: since $\omega_\Gamma^{\log}$ has top degree and because of skew-symmetry of products of 1-forms, the singular logarithmic differential $\mathrm{d}\rho_i/\rho_i$ must be always paired with $\rho_i\,\mathrm{d}\varphi_i$, coming from the complex-valued, real analytic parts of the factors of $\omega_\Gamma^{\log}$. The polar coordinates appear naturally by choosing a global section of the trivial principal $G_3 = \mathbb{R}^+ \ltimes \mathbb{C}$-bundle Conf_n of the configuration space of n points in $\mathbb{C}$, $n \geqslant 2$, which identifies it with $S^1 \times \mathrm{Conf}_{n-2}(\mathbb{C}\setminus\{0,1\})$: the angle coordinates are associated with the S^1-factors and the strata are recovered by setting the radius coordinates to 0. The detailed discussion of this topic can be found in the proof of [2, Proposition 5.2].

These arguments can be slightly adapted to $\omega_\Gamma^{\log,+,-}$, for Γ admissible of type (n,k,l) and $|E(\Gamma)| = 2n+k+l-1$, where $\omega_{\log}$ is replaced by $\omega_{\log}^{+,-}$: namely, $\omega_{\log}^{+,-}$ on $C^+_{n,k,l}$ extends to a complex-valued, real analytic form of top degree on $\bar{C}^+_{n,k,l}$.

Similar arguments imply that $\omega_\Gamma^{\log}$ or $\omega_\Gamma^{\log,+,-}$, for Γ admissible of type (n,m) or (n,k,l) and $|E(\Gamma)| = 2n+m-3$ or $|E(\Gamma)| = 2n+k+l-2$, yield complex-valued forms on $\bar{C}^+_{n,m}$ or $\bar{C}^+_{n,k,l}$ with poles of order 1 along the boundary. Moreover, their formal regularizations along boundary strata of codimension 1 extend to complex-valued, real analytic forms of top degree on those boundary strata: the regularization morphism here formally sets to 0 the logarithmic differentials $\mathrm{d}\rho_i/\rho_i$, whenever $\rho_i = 0$. The detailed version of these arguments can be found in [2, Proposition 5.3].

Theorem 3.2. *Let X be a compact, oriented manifold with corners of degree $d \geqslant 2$. Further, consider an element ω of $\Omega_1^{d-1}(X)$, which satisfies the two additional properties*:

(i) *its exterior derivative $\mathrm{d}\omega$ is a complex-valued, real analytic form of top degree on X, and*
(ii) *the regularization $\mathrm{Reg}_{\partial X}(\omega)$ along the boundary strata ∂X of codimension 1 of X is a complex-valued, real analytic form on ∂X.*

Then, the integrals of $\mathrm{d}\omega$ over X and the integral of $\mathrm{Reg}_{\partial X}(\omega)$ over ∂X exist and the following identity holds true:

$$\int_X \mathrm{d}\omega = \int_{\partial X} \mathrm{Reg}_{\partial X}(\omega).$$

In the assumptions of Theorem 3.2, $\Omega_1^{d-1}(X)$ denotes the space of differential forms ω on X of degree $d-1$ which have the form

$$\omega=\sum_{i=1}^{p}\frac{\mathrm{d}x_i}{x_i}\omega_i+\eta$$

in every local chart of X for which $X=(\mathbb{R}_+)^p\times\mathbb{R}^q$, $p+q=d$, and ω_i, η, $i=1,\dots,p$, are complex-valued, real analytic forms on X. The proof of Theorem 3.2, as well as of other variants of Stokes' Theorem in presence of singularities, can be found in [1, Section 2.3].

Since $\omega_\Gamma^{\log}$ is closed and because of the previous arguments, Stokes' Theorem 3.2 applies to $\omega_\Gamma^{\log}$, whence the associativity of $\star_{\log}$ (more generally, the L_∞-relations for the logarithmic formality quasi-isomorphism and its corresponding version in presence of two branes).

3.2. *Relationship between $\star$, $\star_{\log}$ and the UEA of $\mathfrak{g}$*

Let us restrict our attention to $X=\mathfrak{g}^*$, for $\mathfrak{g}$ as in Section 2: standard arguments imply that the two products (1) restrict to associative products $\star$ and $\star_{\log}$ on $A=\mathbb{S}(\mathfrak{g})$.

Theorem 3.3. *For $\mathfrak{g}$ as in Section 2, there exist isomorphisms of associative algebras $\mathcal{I}$ and $\mathcal{I}_{\log}$ from $(A,\star)$ and $(A,\star_{\log})$ respectively to $(\mathrm{U}(\mathfrak{g}),\cdot)$, which are explicitly given by*

$$\mathcal{I}=\mathrm{PBW}\circ\sqrt{j(\bullet)},\qquad \mathcal{I}_{\log}=\mathrm{PBW}\circ j_\Gamma(\bullet),\tag{2}$$

where $\sqrt{j(\bullet)}$ and $j_\Gamma(\bullet)$ are elements of $\widehat{\mathrm{S}}(\mathfrak{g}^*)$ defined via

$$\sqrt{j(x)}=\sqrt{\det\nolimits_{\mathfrak{g}}\left(\frac{1-e^{-\mathrm{ad}(x)}}{\mathrm{ad}(x)}\right)}=\exp\left(-\frac{1}{4}c_1(x)+\sum_{n\geqslant 1}\frac{\zeta(2n)}{(2n)(2\pi i)^{2n}}c_{2n}(x)\right),\tag{3}$$

$$j_\Gamma(x)=\exp\left(-\frac{1}{4}c_1(x)+\sum_{n\geqslant 2}\frac{\zeta(n)}{n(2\pi i)^n}c_n(x)\right)=\sqrt{j(x)}\exp\left(\sum_{n\geqslant 1}\frac{\zeta(2n+1)}{(2n+1)(2\pi i)^{2n+1}}c_{2n+1}(x)\right),\quad x\in\mathfrak{g},\tag{4}$$

where both elements of the completed symmetric algebra $\widehat{\mathrm{S}}(\mathfrak{g}^*)$ are regarded as invertible differential operators with constant coefficients and of infinite order on A. (We will comment at the end of the proof on the (improperly) adopted notation for both expressions (3) and (4).)

The detailed proof of Theorem 3.3 will be presented in [7]. As it relies heavily on A_∞-algebras and -bimodules, we present here a very quick recall of the definitions for the sake of comprehension.

Digression B. A very quick review of A_∞-structures

Let C be a graded vector space over $\mathbb{K}$: C is called an A_∞-algebra, if the coassociative tensor coalgebra $\mathrm{T}(C[1])$ cofreely cogenerated by $C[1]$ ($[\bullet]$ being the degree-shifting functor on graded vector spaces) with counit admits a coderivation d_C of degree 1, whose square vanishes. Similarly, given two A_∞-algebras (C,d_C), (E,d_E) over $\mathbb{K}$, a graded vector space M over $\mathbb{K}$ is an A_∞–C–E-bimodule, if the cofreely cogenerated bi-comodule $\mathrm{T}(C[1])\otimes M[1]\otimes\mathrm{T}(E[1])$ with natural left- and right-coactions is endowed with a bi-coderivation d_M, whose square vanishes.

Observe that, in view of the cofreeness of $\mathrm{T}(C[1])$ and $\mathrm{T}(C[1])\otimes M[1]\otimes\mathrm{T}(E[1])$, to specify d_C, d_E and d_M is equivalent to specify their Taylor components $d_C^n: C[1]^{\otimes n}\to C[1]$, $d_E^n: E[1]^{\otimes n}\to E[1]$, $n\geqslant 1$, and $d_M^{k,l}: C[1]^{\otimes k}\otimes M[1]\otimes E[1]^{\otimes l}\to M[1]$, $k,l\geqslant 0$, all of degree 1: the condition that d_C, d_E and d_M square to 0 is equivalent to an infinite family of quadratic identities between the respective Taylor components.

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