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Sharp large deviations for some hyperbolic flows

Larges déviations exactes pour certains flots hyperboliques

Vesselin Petkov [a], Luchezar Stoyanov [b]

[a] *Université Bordeaux I, institut de mathématiques, 351, cours de la Libération, 33405 Talence, France*
[b] *University of Western Australia, School of Mathematics and Statistics, Perth, WA 6009, Australia*

A R T I C L E   I N F O

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A B S T R A C T

We prove a sharp large deviation principle concerning intervals shrinking with sub-exponential speed for certain models involving the Poincaré map related to a Markov family for an Axiom A flow restricted to a basic set satisfying some additional regularity assumptions.

R É S U M É

On justifie le principe de larges déviations exactes avec des intervalles décroissants sub-exponentiellement pour certains modèles concernant l'application de Poincaré associée à une famille de Markov pour un Axiom A flot restreint à un ensemble basique qui satisfait des conditions de régularité additionnelles.

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Version française abrégée

Récemment, Pollicott et Sharp [6] ont obtenu un résultat de larges déviations pour des intervalles $(p-\delta_n, p+\delta_n)$ avec $\delta_n \to 0$ quand $n \to \infty$ dans le cas d'un difféomorphisme hyperbolique $f : X \longrightarrow X$. Soit Ψ une fonction continue Hölderienne qui satisfait certaines conditions Diophantiennes associées aux trois orbites périodiques de f. Supposons que l'état d'équilibre m_Ψ de Ψ n'est pas la mesure d'entropie maximale de f et que la suite $\{\delta_n\}$ de nombres positifs est telle que $1/\delta_n = O(n^{1+\kappa})$, $n \to \infty$ avec $\kappa > 0$. Alors dans [6] on prouve que pour tout p dans l'intervalle (1) on a la limite (2). On se propose dans cette Note d'obtenir des exemples quand la limite (2) est valable dans le cas quand $\delta_n \to 0$ avec une vitesse *sous-exponentielle*, c'est-à-dire quand (4) est satisfait. De plus, dans le Théorème 1 nous démontrons que pour les fonctions qu'on étudie, si δ_n satisfait (5) avec $\alpha_0 > 0$ suffisament petit, l'asymptotique (2) n'est pas valable et nous avons la limite (6). A notre connaissance il semble que c'est le premier résultat de larges déviations exactes avec une limite précise différente de la fonction « rate » $-J(p)$ déterminée par (7).

Soit $\varphi_t : M \longrightarrow M$ un C^2 Axiom A flot sur une variété Riemannienne M et soit Λ un ensemble basique de φ_t. Dans notre modèle le rôle de X est joué par l'union de rectangles $R_i = [U_i, S_i]$ d'une famille de Markov $R = \{R_i\}_{i=1}^k$ (cf. [1]). Soit $\mathcal{P} : R \longrightarrow R$ et $\tau : R \longrightarrow [0, \infty)$ l'application de Poincaré et le *premier temps de retour* respectifs tels qu'on a $\varphi_{\tau(x)}(x) = \mathcal{P}(x)$. Dans le modèle qu'on examine, l'application $\mathcal{P}$ joue le rôle de f. Etant donnée une fonction continue Hölderienne $F : \Lambda \longrightarrow \mathbb{R}$ et une fonction Lipschitzienne $G : \Lambda \longrightarrow \mathbb{R}^+$, on introduit $\Psi(x) = G^{\tau(x)} = \int_0^{\tau(x)} G(\varphi_t(x))\,dt$ et $\Phi(x) = F^{\tau(x)} = \int_0^{\tau(x)} F(\varphi_t(x))\,dt$ et pour une fonction h sur R et un entier $n > 1$ on pose $h^n(x) = h(x) + h(\mathcal{P}(x)) + \cdots + h(\mathcal{P}^{n-1}(x))$, $x \in R$.

E-mail addresses: petkov@math.u-bordeaux1.fr (V. Petkov), luchezar.stoyanov@uwa.edu.au (L. Stoyanov).

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On peut représenter φ_t sur Λ par le flot suspendu sur l'ensemble suspendu $R_\tau = \{(x,t)\colon\ x \in R,\ 0 \leqslant t < \tau(x)\}$ (cf. Ch. 6 dans [4]). On suppose que la représentation de G sur R_τ est constante sur les foliations stables, c'est-à-dire sur chaque ensemble de la forme $\{([x,y],t)\colon\ y \in S_i\}$ (voir Sect. 1 pour la définition de $[x,y]$), où $i = 1,\dots,k$, $x \in U_i$ et $t \in [0,\tau(x)]$. De plus, on suppose que $\frac{\mathrm{Lip}(G)}{\min G} \leqslant \mu_0$ avec une petite constante $\mu_0 > 0$.

Sous certaines hypothèses de régularité du flot φ_t (voir les conditions (A), (B), (C) dans Sect. 1) notre résultat principal est le Théorème 1 dans lequel on établit la limite (6) en supposant que δ_n satisfait (5). La preuve du Théorème 1 est basée sur des estimations des itérations de l'opérateur de transfert de Ruelle $\mathcal{L}_{\Phi+(\xi+\mathbf{i}u)\Psi}$ associé à $\Phi + (\xi + \mathbf{i}u)\Psi$, $\xi, u \in \mathbb{R}$, démontrées dans le Théorème 2. Les preuves détailées de nos résultats sont incluses dans [5].

1. Introduction

It follows from general large deviation principles (see [3,11]) that if X is a mixing basic set for an Axiom A diffeomorphism f, Φ and Ψ are Hölder continuous functions on X with equilibrium states m_Φ and m_Ψ, respectively, and m_Ψ is not the measure of maximal entropy of f on X, then there exists a real-analytic *rate function* $J : \mathrm{Int}(\mathcal{I}_\Psi) \longrightarrow [0,\infty)$, where

$$\mathcal{I}_\Psi = \left\{ \int \Psi \, \mathrm{d}m\colon\ m \in \mathcal{M}_X \right\}, \tag{1}$$

$\mathcal{M}_X$ being the set of all f-invariant Borel probability measures on X, such that

$$\lim_{\delta\to 0}\lim_{n\to\infty} \frac{1}{n} \log m_\Phi\left(\left\{ x \in X\colon\ \frac{\Psi^n(x)}{n} \in (p-\delta, p+\delta) \right\} \right) = -J(p), \quad \forall p \in \mathrm{Int}(\mathcal{I}_\Psi). \tag{2}$$

Since m_Ψ is not the measure of maximal entropy, Ψ is not cohomologous to a constant and $\mathrm{Int}(\mathcal{I}_\Psi) \neq \emptyset$. Moreover, $J(p) = 0$ if and only if $p = \int \Psi \, \mathrm{d}m_\Phi$.

Many results on large deviations for hyperbolic (discrete and continuous) dynamical systems have been established in both the uniformly hyperbolic case and the non-uniformly hyperbolic case (see [3,11,6,7,10] and the references given there). Recently, Pollicott and Sharp [6] obtained a large deviation result for shrinking intervals $(p-\delta_n, p+\delta_n)$ with $\delta_n \to 0$ as $n \to \infty$ in the case of a hyperbolic diffeomorphism $f : X \longrightarrow X$. Assuming that the Hölder continuous function Ψ satisfies a certain Diophantine condition related to three periodic orbits of f, the equilibrium state m_Ψ of Ψ is not the measure of maximal entropy of f, and the sequence $\{\delta_n\}$ of positive numbers is such that $1/\delta_n = O(n^{1+\kappa})$ as $n \to \infty$ for some appropriately chosen $\kappa > 0$, they proved that

$$\lim_{n\to\infty} \frac{1}{n} \log m_\Phi\left(\left\{ x \in X\colon\ \frac{\Psi^n(x)}{n} \in (p-\delta_n, p+\delta_n) \right\} \right) = -J(p) \tag{3}$$

for all $p \in \mathrm{Int}(\mathcal{I}_\Psi)$. As a consequence they derived a fluctuation theorem in a similar setup.

Our aim in this Note is to obtain a class of examples where this holds in the case when $\delta_n \to 0$ with *sub-exponential speed*, i.e. when

$$\lim_{n\to\infty} \frac{\log \delta_n}{n} = 0. \tag{4}$$

We also show that for the class of functions we deal with, if $\lim_{n\to\infty} \frac{\log\delta_n}{n} = -\alpha_0$ for some sufficiently small $\alpha_0 > 0$, the asymptotic (3) is not true and we have a lower bound $-J(p) - \alpha_0$. Thus our result in this situation is optimal. To our best knowledge, it seems that this is the first result with a precise limit different from $-J(p)$.

Unlike [6], in our model the role of X is played by the union of all rectangles in a Markov family for a flow restricted to a basic set Λ and f is just the corresponding Poincaré map. Let $\varphi_t : M \longrightarrow M$ $(t \in \mathbb{R})$ be a C^2 Axiom A flow on a Riemannian manifold M and let Λ be a basic set for φ_t. It follows from [1] (see also [2]) that there exists a Markov family $R = \{R_i\}_{i=1}^k$ of rectangles $R_i = [U_i, S_i]$ of arbitrarily small size $\chi > 0$ for the restriction of the flow φ_t to Λ. Let $\mathcal{P} : R \longrightarrow R$ and $\tau : R \longrightarrow [0,\infty)$ be the corresponding Poincaré map and *first return time*, respectively, so that $\varphi_{\tau(x)}(x) = \mathcal{P}(x)$. The *shift map* $\sigma : U = \bigcup_{i=1}^k U_i \longrightarrow U$ is given by $\sigma = \pi^{(U)} \circ \mathcal{P}$, where $\pi^{(U)} : R \longrightarrow U$ is the *projection* along stable leaves. We can model φ_t on Λ by using the so-called suspended flow on the suspension set $R_\tau = \{(x,t)\colon\ x \in R, 0 \leqslant t \leqslant \tau(x)\}$ (see e.g. Ch. 6 in [4]).

Let $F, G : \Lambda \longrightarrow \mathbb{R}$ be Hölder continuous functions. Throughout this Note we assume the following:

Standing assumptions.

(A) φ_t is a mixing flow on a basic set Λ, φ_t and Λ satisfy the conditions LNIC, (R_1) and (R_2) stated below and the local holonomy maps along stable laminations through Λ are uniformly Lipschitz.

(B) $R = \{R_i\}_{i=1}^k$ is a fixed Markov family of rectangles $R_i = [U_i, S_i]$ for the restriction of the flow φ_t to Λ, chosen so that the matrix $\mathcal{A} = \{a_{i,j}\}_{i,j=1}^k$ related to R is irreducible.

(C) $F : \Lambda \longrightarrow \mathbb{R}$ is a Hölder continuous function, while $G : \Lambda \longrightarrow \mathbb{R}$ is Lipschitz and its representative in the suspension space R_τ is constant on stable leaves, i.e. on each set of the form $\{([x,y],t)\colon\ y \in S_i\}$, where $i = 1,\dots,k$, $x \in U_i$ and $t \in [0,\tau(x)]$.

For $T \geqslant 0$ and $x \in \Lambda$ set $G^T(x) = \int_0^T G(\varphi_t(x))\,dt$ and let $\mathcal{I}_0 = \{\int_R G^{\tau(x)}(x)\,dm(x)\colon\ m \in \mathcal{M}_{\mathcal{P}}\}$, where $\mathcal{M}_{\mathcal{P}}$ is the set of all $\mathcal{P}$-invariant Borel probability measures on R. For any function h on R, $x \in R$ and an integer $n \geqslant 1$ we set $h^n(x) = h(x) + h(\mathcal{P}(x)) + \cdots + h(\mathcal{P}^{n-1}(x))$. A Hölder continuous function $g(x)$ on R is called *non-lattice* if there do not exist constant a, a Hölder continuous function h on R and a bounded integer-valued function Z on R so that $g(x) = (h \circ \mathcal{P})(x) - h(x) + a + Z(x)$ for all $x \in R$.

Under the standing assumptions above, in this paper we prove the following main result:

Theorem 1. *There exists a constant $\mu_0 > 0$ such that for any Lipschitz function $G > 0$ on Λ with $\frac{\mathrm{Lip}(G)}{\min G} \leqslant \mu_0$ for which $G^{\tau(x)}(x)$ is a non-lattice function on R there exists a rate function $J : \mathcal{I}_0 \longrightarrow [0, \infty)$ with the following property: for every Hölder continuous function F on Λ there exists a constant $\rho = \rho(F, G) \in (0, 1)$ such that for any sequence $\{\delta_n\}$ of positive numbers decreasing to zero with*

$$\lim_{n\to\infty} \frac{\log \delta_n}{n} = -\alpha_0 \leqslant 0 \tag{5}$$

for some $0 \leqslant \alpha_0 \leqslant -\frac{\log \rho}{2}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \mu\left(\left\{x \in R\colon\ \frac{G^{\tau^n(x)}(x)}{n} \in (p - \delta_n, p + \delta_n)\right\}\right) = -J(p) - \alpha_0, \quad \forall p \in \mathrm{Int}(\mathcal{I}_0), \tag{6}$$

where μ is the equilibrium state of the function $\Phi(x) = F^{\tau(x)}(x)$ on R.

The rate function J is explicitly defined in Section 2 below. It is easy to see that there is a non-trivial open set of (essentially) Lipschitz functions $\Psi(x) = G^{\tau(x)}(x)$ on R for which Theorem 1 applies. More precisely, for any constant $c > 0$ there is an open neighborhood V of $c\tau$ in the space of (essentially) Lipschitz functions on R constant on stable leaves such that for any $\Psi \in V$ Theorem 1 applies. What concerns the standing assumptions, various classes of flows on basic sets satisfying these are described in [8,9] (see also the references there).

Let $W^s_{\epsilon_0}(x)$ and $W^u_{\epsilon_0}(x)$ be the strong stable and unstable manifolds of size ϵ_0 through $x \in \Lambda$. It follows from the hyperbolicity of Λ that if $\epsilon_0 > 0$ is sufficiently small, there exists $\epsilon_1 > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \epsilon_1$, then $W^s_{\epsilon_0}(x)$ and $\varphi_{[-\epsilon_0,\epsilon_0]}(W^u_{\epsilon_0}(y))$ intersect at exactly one point $[x, y] \in \Lambda$. That is, there exists a unique $t \in [-\epsilon_0, \epsilon_0]$ such that $\varphi_t([x, y]) \in W^u_{\epsilon_0}(y)$. Setting $\Delta(x, y) = t$ defines the so-called *temporal distance function*. For $x, y \in \Lambda$ with $d(x, y) < \epsilon_1$, define $\pi_y(x) = [x, y]$. Notice that τ is constant on each stable leaf $W^s_{R_i}(x) = W^s_{\epsilon_0}(x) \cap R_i$.

The following *local non-integrability condition* for φ_t and Λ was introduced in [8].

LNIC. There exist $z_0 \in \Lambda$, $\epsilon_0 > 0$ and $\theta_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$, any $\hat{z} \in \Lambda \cap W^u_\epsilon(z_0)$ and any tangent vector $\eta \in E^u(\hat{z})$ to Λ at $\hat{z}$ with $\|\eta\| = 1$ there exist $\tilde{z} \in \Lambda \cap W^u_\epsilon(\hat{z})$, $\tilde{y}_1, \tilde{y}_2 \in \Lambda \cap W^s_\epsilon(\tilde{z})$ with $\tilde{y}_1 \neq \tilde{y}_2$, $\delta > 0$ and $\epsilon' > 0$ such that

$$\left|\Delta\big(\exp^u_z(v), \pi_{\tilde{y}_1}(z)\big) - \Delta\big(\exp^u_z(v), \pi_{\tilde{y}_2}(z)\big)\right| \geqslant \delta \|v\|$$

for all $z \in W^u_{\epsilon'}(\tilde{z}) \cap \Lambda$ and $v \in E^u(z; \epsilon')$ with $\exp^u_z(v) \in \Lambda$ and $\langle \frac{v}{\|v\|}, \eta_z \rangle \geqslant \theta_0$, where η_z is the parallel translate of η along the geodesic in $W^u_{\epsilon_0}(z_0)$ from $\hat{z}$ to z.

Set

$$B^u_T(x, \epsilon) = \big\{y \in W^u_\epsilon(x) \cap \Lambda\colon\ d\big(\varphi_t(x), \varphi_t(y)\big) \leqslant \epsilon, 0 \leqslant t \leqslant T\big\}.$$

Following [8], we will say that ϕ_t has a *regular distortion along unstable manifolds* over the basic set Λ if there exists a constant $\epsilon_0 > 0$ with the following properties:

- (R_1) For any $0 < \delta \leqslant \epsilon \leqslant \epsilon_0$ there exists a constant $R = R(\delta, \epsilon) > 0$ such that $\mathrm{diam}(\Lambda \cap B^u_T(z, \epsilon)) \leqslant R\, \mathrm{diam}(\Lambda \cap B^u_T(z, \delta))$ for any $z \in \Lambda$ and any $T > 0$.
- (R_2) For any $\epsilon \in (0, \epsilon_0]$ and any $\rho \in (0, 1)$ there exists $\delta \in (0, \epsilon]$ such that for any $z \in \Lambda$ and any $T > 0$ we have $\mathrm{diam}(\Lambda \cap B^u_T(z, \delta)) \leqslant \rho\, \mathrm{diam}(\Lambda \cap B^u_T(z, \epsilon))$.

2. Ruelle transfer operator

Given a Markov family $\mathcal{R}$ as in Section 1, denote by $\widehat{R}$ the *core* of R, i.e. the set of those $x \in R$ such that $\mathcal{P}^m(x) \in \mathrm{Int}_\Lambda(R)$ for all $m \in \mathbb{Z}$. It is well-known (see [1]) that $\hat{R}$ is a residual subset of R and has full measure with respect to any Gibbs measure on R. The set $\hat{U} = U \cap \hat{R}$ has similar properties. In general τ is not continuous on U, however, under the standing assumption (A), τ is *essentially Lipschitz* on U in the sense that there exists a constant $L > 0$ such that if $x, y \in U_i \cap \sigma^{-1}(U_j)$ for some i, j, then $|\tau(x) - \tau(y)| \leqslant L\, d(x, y)$. In a similar way one defines essentially Lipschitz functions on R. Let $\mathrm{Pr}_{\mathcal{P}}(h)$ be the *topological pressure* of a continuous function h on R with respect to the map $\mathcal{P}$ on R (see e.g. [4]).

Let $F, G : \longrightarrow \mathbb{R}$ be Hölder continuous functions, and let $\Phi(x) = F^{\tau(x)}(x)$ and $\Psi(x) = G^{\tau(x)}(x)$ $(x \in R)$. It is easy to check that $\Psi^n(x) = G^{\tau^n(x)}(x)$ for all $x \in R$. It follows from the Large Deviation Theorem in [3] that if m_Ψ is not the measure of maximal entropy for $\mathcal{P}$, then there exists a real-analytic function $J : \mathrm{Int}(\mathcal{I}_\Psi) \longrightarrow [0, \infty)$ such that $J(p) = 0$ iff $p = \int_R \Psi \, \mathrm{d}m_\Phi$ for which (2) holds. More precisely, we have

$$-J(p) = \inf\{\Pr_{\mathcal{P}}(\Phi + q\Psi) - \Pr_{\mathcal{P}}(\Phi) - qp \colon q \in \mathbb{R}\}. \tag{7}$$

Let $B(\hat{U})$ be the *space of bounded functions* $g : \hat{U} \longrightarrow \mathbb{C}$ with its standard norm $\|g\|_\infty = \sup_{x \in \hat{U}} |g(x)|$. Given a function $g \in B(\hat{U})$, the *Ruelle transfer operator* $\mathcal{L}_g : B(\hat{U}) \longrightarrow B(\hat{U})$ is defined by $(\mathcal{L}_g h)(u) = \sum_{\sigma(v)=u} e^{g(v)} h(v)$. If $g \in B(\hat{U})$ is essentially Lipschitz on $\hat{U}$, then $\mathcal{L}_g$ preserves the space $C^{\mathrm{Lip}}(\hat{U})$ of *Lipschitz functions* $g : \hat{U} \longrightarrow \mathbb{C}$. Let $\mathrm{Lip}(g)$ be the Lipschitz constant of g.

Given a Lipschitz real-valued function f on $\hat{U}$, set $\tilde{f} = f - P\Psi$, where $P = P_f \in \mathbb{R}$ is the unique number such that the topological pressure $\Pr_\sigma(\tilde{f})$ of $\tilde{f}$ with respect to σ is zero (cf. e.g. [4]). For $a, b \in \mathbb{R}$, consider the *Ruelle transfer operator* $\mathcal{L}_{\tilde{f}-(a+\mathbf{i}b)\Psi} : C^{\mathrm{Lip}}(\hat{U}) \longrightarrow C^{\mathrm{Lip}}(\hat{U})$. We will say that the *Ruelle transfer operators related to Ψ and the function f on $\hat{U}$ are eventually contracting* if for every $\epsilon > 0$ there exist constants $0 < \rho < 1$, $a_0 > 0$ and $C > 0$ such that if $a, b \in \mathbb{R}$ satisfy $|a| \leqslant a_0$ and $|b| \geqslant 1/a_0$, then for every integer $m > 0$ and every $h \in C^{\mathrm{Lip}}(\hat{U})$ we have

$$\left\| \mathcal{L}^m_{f-(P_f+a+\mathbf{i}b)\Psi} h \right\|_{\mathrm{Lip},b} \leqslant C\rho^m |b|^\epsilon \|h\|_{\mathrm{Lip},b},$$

where the norm $\|.\|_{\mathrm{Lip},b}$ on $C^{\mathrm{Lip}}(\hat{U})$ is defined by $\|h\|_{\mathrm{Lip},b} = \|h\|_\infty + \frac{\mathrm{Lip}(h)}{|b|}$. This implies in particular that the spectral radius of $\mathcal{L}_{f-(P_f+a+\mathbf{i}b)\Psi}$ on $C^{\mathrm{Lip}}(\hat{U})$ does not exceed ρ.

The following theorem is one of the main ingredients in the proof of Theorem 1:

Theorem 2. *Under the standing assumptions, let $\Psi(x) = G^{\tau(x)}(x)$ $(x \in R)$. Then there exists a constant $\mu_0 > 0$ such that if $G > 0$ and $\frac{\mathrm{Lip}(G)}{\min G} \leqslant \mu_0$, then for any Hölder continuous real-valued function f on $\hat{U}$ the Ruelle transfer operators related to Ψ and f are eventually contracting.*

Consider a sequence $\{\delta_n\}_{n \in \mathbb{N}}$, $\delta_n > 0$, $\delta_n \to 0$, such that (5) holds and let $\epsilon_n = n\delta_n$. Fix an arbitrary $p \in \mathrm{Int}(\mathcal{I}_\Psi)$ and set $\Psi_p = \Psi - p$. As in [6], it is enough to prove a modified result concerning a sequence of the form $\rho(n) = \int_U \chi_n(\Psi_p^n(x)) \, \mathrm{d}\mu_\Phi$, where $\chi \in C_0^k(\mathbb{R} : \mathbb{R})$ is a *fixed cut-off function* and $\chi_n(x) = \chi(\epsilon_n^{-1} x)$ for $x \in \mathbb{R}$.

Proposition 3. *Under the assumptions of Theorem* 2*, we have* $\lim_{n\to\infty} \frac{1}{n} \log \rho(n) = -J(p) - \alpha_0$.

Theorem 1 follows immediately from Proposition 3 as shown in [6].

3. Idea of the proof of Theorem 2

Define the *temporal Ψ-function* Δ_Ψ by

$$\Delta_\Psi(x, y) = \int_0^{\Delta(x,y)} G(\varphi_t([x, y])) \, \mathrm{d}t$$

for $x, y \in \Lambda$, $d(x, y) < \epsilon_1$. Just like Δ, this function is constant on stable leaves with respect to the first variable and constant on unstable leaves with respect to the second.

Given a Lipschitz real-valued function f on $\hat{U}$, let again $\tilde{f} = f - P\Psi$, where $P = P_f$. To prove Theorem 2 we apply the arguments from Sections 3 and 5 in [8] and a modification of the arguments in Section 4 there. The main step is to prove the analogue of Lemma 4.2 in [8] with the function τ replaced by Ψ.

Following Section 4 in [8], **fix an arbitrary point** $z_0 \in \Lambda$ **and constants** $\epsilon_0 > 0$ **and** $\theta_0 \in (0, 1)$ **with the properties described in LNIC**. Assume that $z_0 \in \mathrm{Int}_\Lambda(U_1)$, $U_1 \subset \Lambda \cap W^u_{\epsilon_0}(z_0)$ and $S_1 \subset \Lambda \cap W^s_{\epsilon_0}(z_0)$.

Next, fix a C^1 parametrization $r(s) = \exp^u_{z_0}(s)$, $s \in V_0'$, of a small neighborhood W_0 of z_0 in $W^u_{\epsilon_0}(z_0)$ such that V_0' is a convex compact neighborhood of 0 in $E^u(z_0)$. Then $r(0) = z_0$ and $\frac{\partial}{\partial s_i} r(s)_{|s=0} = e_i$ for all $i = 1, \dots, n$. Set $U_0' = W_0 \cap \Lambda$, and let $\theta < \theta_1 < 1$.

Following [8], for a cylinder $\mathcal{C} \subset U_0'$ and a unit vector $\xi \in E^u(z_0)$ we will say that a *separation by a ξ-plane occurs* in $\mathcal{C}$ if there exist $u, v \in \mathcal{C}$ with $d(u, v) \geqslant \frac{1}{2} \mathrm{diam}(\mathcal{C})$ such that $\langle \frac{r^{-1}(v) - r^{-1}(u)}{\|r^{-1}(v) - r^{-1}(u)\|}, \xi \rangle \geqslant \theta_1$.

Let $\mathcal{S}_\xi$ be the *family of all cylinders* $\mathcal{C}$ contained in U_0' such that a separation by a ξ-plane occurs in $\mathcal{C}$. Given an open subset V of U_0' which is a finite union of open cylinders and $\delta > 0$, let $\mathcal{C}_1, \dots, \mathcal{C}_p$ $(p = p(\delta) \geqslant 1)$ be the family of maximal closed cylinders in $\bar{V}$ with $\mathrm{diam}(\mathcal{C}_j) \leqslant \delta$. For any unit vector $\xi \in E^u(z_0)$ set $M^{(\delta)}_\xi(V) = \bigcup\{\mathcal{C}_j \colon \mathcal{C}_j \in \mathcal{S}_\xi, 1 \leqslant j \leqslant p\}$.

Given a sequence of unit vectors $\xi_1, \xi_2, \ldots, \xi_{j_0} \in E^u(z_0)$, set $B_\ell = \{\eta \in \mathbf{S}^{n-1}: \langle \eta, \xi_\ell \rangle \geqslant \theta_0\}$ for each $\ell = 1, \ldots, j_0$. For $t \in \mathbb{R}$ and $s \in E^u(z_0)$ set $I_{\eta,t} g(s) = \frac{g(s+t\eta)-g(s)}{t}$, $t \neq 0$.

The main aim is to prove the following analogue of Lemma 4.2 in [8]:

Lemma 4. *There exist integers $1 \leqslant n_1 \leqslant N_0$ and $\ell_0 \geqslant 1$, a sequence of unit vectors $\eta_1, \eta_2, \ldots, \eta_{\ell_0} \in E^u(z_0)$ and a non-empty open subset U_0 of U'_0 which is a finite union of open cylinders of length n_1 such that setting $\mathcal{U} = \sigma^{n_1}(U_0)$ we have:*

(a) *For any integer $N \geqslant N_0$ there exist Lipschitz maps $v_1^{(\ell)}, v_2^{(\ell)} : U \longrightarrow U$ $(\ell = 1, \ldots, \ell_0)$ such that $\sigma^N(v_i^{(\ell)}(x)) = x$ for all $x \in \mathcal{U}$ and $v_i^{(\ell)}(\mathcal{U})$ is a finite union of open cylinders of length N $(i = 1, 2;\ \ell = 1, 2, \ldots, \ell_0)$.*

(b) *There exists a constant $\hat{\delta} > 0$ such that for all $\ell = 1, \ldots, \ell_0$, $s \in r^{-1}(U_0)$, $0 < |h| \leqslant \hat{\delta}$ and $\eta \in B_\ell$ so that $s + h\eta \in r^{-1}(U_0 \cap \Lambda)$ we have*

$$\left[I_{\eta,h}\left(\Psi^N\left(v_2^{(\ell)}\left(\tilde{r}(\cdot)\right)\right) - \Psi^N\left(v_1^{(\ell)}\left(\tilde{r}(\cdot)\right)\right)\right)\right](s) \geqslant A\hat{\delta}/4.$$

(c) *For any open cylinder V in U_0 there exists $\delta' > 0$ such that $V \subset M_{\eta_1}^{(\delta)}(V) \cup \cdots \cup M_{\eta_{\ell_0}}^{(\delta)}(V)$ for all $\delta \in (0, \delta']$.*

Using the objects constructed in Lemma 4.4 in [8] and essentially repeating some of the arguments from the proof of this lemma, one gets an analogue of the lemma, where the function τ is replaced by Ψ, from which Lemma 4 above is derived using again arguments similar to these in Section 4 in [8].

Proof of Theorem 2. Once Lemma 3 is proved, for Lipschitz functions f the rest of the argument is just a repetition of Section 5 in [8] without any changes. For Hölder continuous f one just needs to combine this with the approximation procedure in [2]. Since Ψ is Lipschitz, the approximation procedure can be carried out in the same way as in [2]. □

The detailed proofs of our results are contained in [5].

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