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Differential Geometry

Log-concavity of complexity one Hamiltonian torus actions

Log-concavité des actions toriques hamiltoniennes de complexité un

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ABSTRACT

Let (M,ω) be a closed $2n$-dimensional symplectic manifold equipped with a Hamiltonian T^{n-1}-action. Then Atiyah–Guillemin–Sternberg convexity theorem implies that the image of the moment map is an $(n-1)$-dimensional convex polytope. In this Note, we show that the density function of the Duistermaat–Heckman measure is log-concave on the image of the moment map.

RÉSUMÉ

Soit (M,ω) une variété symplectique de dimension $2n$ munie d'une action hamiltonienne du tore T^{n-1}. Le théorème de convexité d'Atiyah–Guillemin–Sternberg implique que l'image de l'application moment est un polytope convexe de dimension $(n-1)$. Dans cette Note, nous montrons que la fonction de densité de la mesure de Duistermaat–Heckman est log-concave sur l'image de l'application moment.

1. Introduction

In statistical physics, the relation $S(E)=k\log W(E)$ is called Boltzmann's principle where W is the number of states with given values of macroscopic parameters E (like energy, temperature, …), k is the Boltzmann constant, and S is the entropy of the system, which measures the degree of disorder in the system. For the additive values E, it is well known that the entropy is always a concave function. (See [9] for more details.) In a symplectic setting, consider a Hamiltonian G-manifold (M,ω) with the moment map $\mu : M \to \mathfrak{g}^*$. The Liouville measure m_L is defined by

$$m_L(U) := \int_U \frac{\omega^n}{n!}$$

for any open set $U \subset M$. Then the push-forward measure $m_{\mathrm{DH}} := \mu_* m_L$, called the *Duistermaat–Heckman measure*, can be regarded as a measure on $\mathfrak{g}^*$ such that for any Borel subset $B \subset \mathfrak{g}^*$, $m_{\mathrm{DH}}(B) = \int_{\mu^{-1}(B)} \frac{\omega^n}{n!}$ tells us that how many states of our system have momenta in B. By the Duistermaat–Heckman theorem [2], m_{DH} can be expressed in terms of the density function $\mathrm{DH}(\xi)$ with respect to the Lebesque measure on $\mathfrak{g}^*$. Therefore the concavity of the entropy of a given periodic Hamiltonian system on (M,ω) can be interpreted as the log-concavity of $\mathrm{DH}(\xi)$ on the image of μ. A. Okounkov [10]

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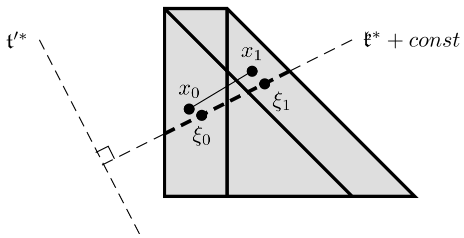

Fig. 1. Proof of Theorem 1.1.

proved that the density function of the Duistermaat–Heckman measure is log-concave on the image of the moment map for the maximal torus action, when (M,ω) is the co-adjoint orbit of some classical Lie groups. In [3], W. Graham showed that the log-concavity of the density function of the Duistermaat–Heckman measure also holds for any Kähler manifold admitting a holomorphic Hamiltonian torus action. V. Ginzberg and A. Knutson conjectured independently that the log-concavity holds for any Hamiltonian G-manifolds, but this turns out to be false in general, shown by Y. Karshon [5]. Further related works can be found in [7] and [1].

As noted in [5] and [3], log-concavity holds for Hamiltonian toric (i.e. complexity zero) actions, and Y. Lin dealt with the log-concavity of complexity two Hamiltonian torus actions in [7]. However, there is no result on the log-concavity of complexity one Hamiltonian torus action. This is why we restrict our interest to complexity one. From now on, we assume that (M,ω) is a $2n$-dimensional closed symplectic manifold with an effective Hamiltonian T^{n-1}-action. Let $\mu : M \to \mathfrak{t}^*$ be the corresponding moment map where $\mathfrak{t}^*$ is a dual of the Lie algebra of T^{n-1}. By the Atiyah–Guillemin–Sternberg convexity theorem, the image of the moment map $\mu(M)$ is an $(n-1)$-dimensional convex polytope in $\mathfrak{t}^*$. By the Duistermaat–Heckman theorem [2], we have

$$m_{\mathrm{DH}} = \mathrm{DH}(\xi)\,\mathrm{d}\xi$$

where $\mathrm{d}\xi$ is the Lebesque measure on $\mathfrak{t}^* \cong \mathbb{R}^{n-1}$ and $\mathrm{DH}(\xi)$ is a continuous piecewise polynomial function of degree less than 2 on $\mathfrak{t}^*$. Our main theorem is as follows:

Theorem 1.1. *Let (M,ω) be a $2n$-dimensional closed symplectic manifold equipped with a Hamiltonian T^{n-1}-action with the moment map $\mu : M \to \mathfrak{t}^*$. Then the density function of the Duistermaat–Heckman measure is log-concave on $\mu(M)$.*

2. Proof of Theorem 1.1

Let (M,ω) be a $2n$-dimensional closed symplectic manifold. Let $(n-1)$-dimensional torus T act on (M,ω) in Hamiltonian fashion. Denote by $\mathfrak{t}$ the Lie algebra of T. For a moment map $\mu : M \to \mathfrak{t}^*$ of the T-action, define the Duistermaat–Heckman function $\mathrm{DH} : \mathfrak{t}^* \to \mathbb{R}$ as

$$\mathrm{DH}(\xi) = \int_{M_\xi} \omega_\xi$$

where M_ξ is the reduced space $\mu^{-1}(\xi)/T$ and ω_ξ is the corresponding reduced symplectic form on M_ξ.

Now, we define the x-ray of our action. Let $T_1, \ldots, T_N$ be the subgroups of T^{n-1} which occur as stabilizers of points in M^{2n}. Let M_i be the set of points whose stabilizers are T_i. By relabeling, we can assume that the M_i's are connected and the stabilizer of points in M_i is T_i. Then, M^{2n} is a disjoint union of M_i's. Also, it is well known that M_i is open dense in its closure and the closure is just a component of the fixed set M^{T_i}. Let $\mathfrak{M}$ be the set of M_i's. Then, the *x-ray* of (M^{2n},ω,μ) is defined as the set of $\mu(\overline{M_i})$'s. Here, we recall a basic lemma:

Lemma 2.1. *(See [4, Theorem 3.6].) Let $\mathfrak{h}$ be the Lie algebra of T_i. Then $\mu(M_i)$ is locally of the form $x + \mathfrak{h}^\perp$ for some $x \in \mathfrak{t}^*$.*

By this lemma, $\dim_{\mathbb{R}} \mu(M_i) = m$ for $(n-1-m)$-dimensional T_i. Each image $\mu(\overline{M_i})$ (resp. $\mu(M_i)$) is called an *m-face* (resp. *an open m-face*) of the x-ray if T_i is $(n-1-m)$-dimensional. Our interest is mainly in open $(n-2)$-faces of the x-ray, i.e. codimension one in $\mathfrak{t}^*$. Fig. 1 is an example of x-ray with $n=3$ where thick lines are $(n-2)$-faces. Now, we can prove the main theorem.

Proof of Theorem 1.1. When $n=2$, we obtain a proof by [6, Lemma 2.19]. So, we assume $n \geqslant 3$. Pick arbitrary two points x_0, x_1 in the image of μ. We should show that

$$t\log\big(\mathrm{DH}(x_1)\big) + (1-t)\log\big(\mathrm{DH}(x_0)\big) \leqslant \log\big(\mathrm{DH}\big(tx_1 + (1-t)x_0\big)\big) \tag{1}$$

for each $t \in [0,1]$. Put $x_t = tx_1 + (1-t)x_0$.

Let us fix a decomposition $T = S^1 \times \cdots \times S^1$. By the decomposition, we identify $\mathfrak{t}$ with $\mathbb{R}^{n-1}$, and $\mathfrak{t}$ carries the usual Riemannian metric $\langle,\rangle_0$ which is a bi-invariant metric. This metric gives the isomorphism

$$\iota : \mathfrak{t} \to \mathfrak{t}^*, \qquad X \mapsto \langle \cdot, X\rangle_0.$$

For a small $\epsilon > 0$, pick two regular values ξ_i in the ball $B(x_i, \epsilon)$ for $i = 0, 1$ which satisfy the following two conditions:

i. $\xi_1 - \xi_0 \in \iota(\mathbb{Q}^{n-1})$,
ii. the line L containing ξ_0, ξ_1 in $\mathfrak{t}^*$ meets each open m-face transversely for $m = 1, \ldots, n-2$.

Transversality guarantees that the line does not meet any open m-face for $m \leqslant n-3$. Put

$$\xi_t = t\xi_1 + (1-t)\xi_0 \quad \text{and} \quad X = \iota^{-1}(\xi_1 - \xi_0).$$

Let $\mathfrak{k} \subset \mathfrak{t}$ be the one-dimensional subalgebra spanned by X. By i., $\mathfrak{k}$ becomes a Lie algebra of a circle subgroup of T, call it K. Let $\mathfrak{t}'$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{t}$. Again by i., $\mathfrak{t}'$ becomes a Lie subgroup of an $(n-2)$-dimensional subtorus of T, call it T'. Let

$$p : \mathfrak{t}^* \to \mathfrak{t}'^* = \iota(\mathfrak{t}')$$

be the orthogonal projection along $\mathfrak{k}^* = \iota(\mathfrak{k}')$. If we put $\mu' = p \circ \mu$, then $\mu' : M \to \mathfrak{t}'^*$ is a moment map of the restricted T'-action on M. Put $\xi' = p(\xi_t)$ for $t \in [0,1]$.

We want to show that ξ' is a regular value of μ'. For this, we show that each point $x \in \mu'^{-1}(\xi')$ is a regular point of μ'. By ii. and Lemma 2.1, stabilizer T_x is finite or one-dimensional. If T_x is finite, then x is a regular point of μ so that it is also a regular point of μ'. If T_x is one-dimensional, then $\mu(x)$ is a point of an open $(n-2)$-face $\mu(M_i)$ such that $x \in M_i$. Let $\mathfrak{h}$ be the Lie algebra of $T_i = T_x$. By Lemma 2.1, $p(d\mu(T_x M_i)) = p(\mathfrak{h}^\perp)$, and the kernel $\mathfrak{k}$ of p is not contained in $\mathfrak{h}^\perp$ by transversality. So, $p(\mathfrak{h}^\perp)$ is the whole $\mathfrak{t}'^*$ because $\dim \mathfrak{h}^\perp = \dim \mathfrak{t}'^*$, and this means that x is a regular point of μ'. Therefore, we have shown that ξ' is a regular value of μ'.

Since ξ' is a regular value, the preimage $\mu'^{-1}(\xi')$ is a manifold and T' acts almost freely on it, i.e. stabilizers are finite. So, if we denote by $M_{\xi'}$ the symplectic reduction $\mu'^{-1}(\xi')/T'$, then it becomes a symplectic orbifold carrying the induced symplectic T/T'-action. We can observe that the image of $\mu'^{-1}(\xi')$ through μ is the thick dashed line in Fig. 1. Since $K/(K \cap T') \cong T/T'$, we will regard $K/(K \cap T')$ and $\mathfrak{k}$ as T/T' and its Lie algebra, respectively. The map $\mu_X := \langle \mu, X\rangle$ induces a map on $M_{\xi'}$ by T-invariance of μ, call it just μ_X where $\langle,\rangle : \mathfrak{t}^* \times \mathfrak{t} \to \mathbb{R}$ is the evaluation pairing. Then, we can observe that μ_X is a Hamiltonian of the $K/(K \cap T')$-action on $M_{\xi'}$, and that M_{ξ_t} is symplectomorphic to the symplectic reduction of $M_{\xi'}$ at the regular value $\langle \xi_t, X\rangle$ with respect to μ_X. If we denote by DH_X the Duistermaat–Heckman function of $\mu_X : M_{\xi'} \to \mathbb{R}$, then we have $\mathrm{DH}(\xi_t) = \mathrm{DH}_X(\langle \xi_t, X\rangle)$ for $t \in [0,1]$. Since $M_{\xi'}$ is a four-dimensional symplectic orbifold with Hamiltonian circle action, DH_X is log-concave by Lemma 2.2 below. Since x_t and ξ_t are sufficiently close and DH is continuous by [2], we can show (1) by log-concavity of DH_X. $\square$

Lemma 2.2. *Let (N, σ) be a closed four-dimensional Hamiltonian S^1-orbifold. Then the density function of the Duistermaat–Heckman measure is log-concave.*

Proof. Let $\phi : N \to \mathbb{R}$ be a moment map. Then the density function $\mathrm{DH} : \mathrm{Im}\,\phi \to \mathbb{R}_{\geqslant 0}$ of the Duistermaat–Heckman measure is given by

$$\mathrm{DH}(t) = \int_{N_t} \sigma_t$$

for any regular value $t \in \mathrm{Im}\,\phi$. Let $(a,b) \subset \mathrm{Im}\,\phi$ be an open interval consisting of regular values of ϕ and fix $t_0 \in (a,b)$. By the Duistermaat–Heckman theorem [2], $[\sigma_t] - [\sigma_{t_0}] = -e(t - t_0)$ for any $t \in (a,b)$, where e is the Euler class of the S^1-fibration $\phi^{-1}(t_0) \to \phi^{-1}(t_0)/S^1$. Therefore

$$\mathrm{DH}'(t) = -\int_{N_t} e$$

and

$$\mathrm{DH}''(t) = 0$$

for any $t \in (a,b)$. Note that $\mathrm{DH}(t)$ is log-concave on (a,b) if and only if it satisfies $\mathrm{DH}(t) \cdot \mathrm{DH}''(t) - \mathrm{DH}'(t)^2 \leqslant 0$ for all $t \in (a,b)$. Hence $\mathrm{DH}(t)$ is log-concave on any open intervals consisting of regular values.

Let c be any interior critical value of ϕ in $\operatorname{Im}\phi$. Then it is enough to show that the jump in the derivative of $(\log \mathrm{DH})'$ is negative at c. First, we will show that the jump of the value $\mathrm{DH}'(t) = -\int_{N_t} e$ is negative at c. Choose a small $\epsilon > 0$ such that $(c-\epsilon, c+\epsilon)$ does not contain a critical value except for c. Let N_c be a symplectic cut of $\phi^{-1}[c-\epsilon, c+\epsilon]$ along the extremum so that N_c becomes a closed Hamiltonian S^1-orbifold whose maximum is the reduced space $M_{c+\epsilon}$ and the minimum is $N_{c-\epsilon}$. Using the Atiyah–Bott–Berline–Vergne localization formula for orbifolds [8], we have

$$0 = \int_{N_c} 1 = \sum_{p \in N^{S^1} \cap \phi^{-1}(c)} \frac{1}{d_p} \frac{1}{p_1 p_2 \lambda^2} + \int_{M_{c-\epsilon}} \frac{1}{\lambda + e_-} + \int_{N_{c+\epsilon}} \frac{1}{-\lambda - e_+}$$

which is equivalent to

$$0 = \sum_{p \in N^{S^1} \cap \phi^{-1}(c)} \frac{1}{p_1 p_2} = \int_{N_{c-\epsilon}} e_- - \int_{N_{c+\epsilon}} e_+,$$

where d_p is the order of the local group of p, p_1 and p_2 are the weights of the tangential S^1-representation on T_pN, and e_- (e_+ respectively) is the Euler class of $\phi^{-1}(c-\epsilon)$ ($\phi^{-1}(c+\epsilon)$ respectively). Since c is in the interior of $\operatorname{Im}\phi$, we have $p_1 p_2 < 0$ for any $p \in N^{S^1} \cap \phi^{-1}(c)$. Hence the jump of $\mathrm{DH}'(t) = -\int_{N_t} e$ is negative at c, which implies that the jump of $\log \mathrm{DH}(t)' = \frac{\mathrm{DH}'(t)}{\mathrm{DH}(t)}$ is negative at c (by continuity of $\mathrm{DH}(t)$). This finishes the proof. □

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