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Numerical Analysis

An asymptotic preserving scheme with the maximum principle for the M_1 model on distorded meshes

Un nouveau schéma préservant l'asymptotique avec le principe du maximum pour le modèle M_1 sur maillages quelconques

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A B S T R A C T

In this Note, we show that a recent scheme introduced by Buet et al. (2011) [5] for the nonlinear two moments M_1 model of linear transport and which captures correctly the diffusion limit on distorded meshes (AP scheme) also possesses the maximum principle. The main idea of the design of this scheme is to rewrite the model as a gas dynamics model and to use an Eulerian scheme, derived from a Lagrange + remap scheme. To obtain the AP property we use the multidimensional extension, developed by Buet et al. (2012) [6], of the Jin and Levermore (1996) procedure [9] for the hyperbolic heat equation. We will show that this scheme is entropic which ensures the maximum principle of the M_1 model. More we present some numerical results, on distorted quadrangular and triangular meshes which show that the scheme is second order in the diffusive regime.

R É S U M É

Dans cette Note, nous montrons qu'un nouveau schéma introduit dans Buet et al. (2011) [5] pour le modèle à deux moments non linéaire M_1 de l'équation de transport et qui est compatible avec la limite de diffusion (schéma AP) sur maillage quelconque vérifie aussi le principe du maximum. L'idée consiste à réécrire le modèle comme un système de la dynamique des gaz, puis à utiliser un schéma Eulerien nodal, dérivé d'un schéma Lagrange + projection couplé à une extension multidimensionnelle, developpée dans Buet et al. (2012) [6], de la méthode de Jin et Levermore (1996) [9] pour l'équation de la chaleur hyperbolique. Après la présentation du schéma on donne les preuves d'entropie et de principe du maximum. Pour finir on présente des résultats numériques pour des maillages déformés triangulaires et quadrangulaires qui montrent notamment l'ordre deux dans le régime de diffusion.

Version française abrégée

L'objectif de cette Note est de démontrer le principe du maximum pour une discrétisation AP du modèle M_1 sur maillages quelconques. À notre connaissance ce schéma est le premier de ce type sur maillages quelconques. La méthode que

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nous présentons est basée sur une reformulation du modèle M_1 sous la forme d'un système proche des équations d'Euler comme dans [1,4] et l'utilisation d'un schéma Lagrange-projection nodal pour l'hydrodynamique [7,11]. La prise en compte du terme source dans le solveur de Riemann par une extension multidimensionnelle de la méthode de Jin–Levermore [9], proposée dans [6] pour le modèle P_1 en 2D sur des maillages généraux, permet d'obtenir un schéma AP. Le schéma ainsi obtenu est un schéma entropique, qui préserve le principe de maximum du modèle M_1 et est consistant et AP sur maillage non structurés. Par une procédure de type MUSCL dans la phase de projection on obtient alors un schéma d'ordre 2 en régime de diffusion.

1. Introduction

In this Note, we are interested in the discretization of the nonlinear M_1 system which arises from an entropy closure of the transfer equation for photons

$$\begin{cases} \partial_t E + \frac{1}{\varepsilon}\nabla.\mathbf{F} = 0, & E \in \mathbb{R} \\ \partial_t \mathbf{F} + \frac{1}{\varepsilon}\nabla.\widehat{P} = -\frac{\sigma}{\varepsilon^2}\mathbf{F}, & \mathbf{F} \in \mathbb{R}^d \end{cases} \tag{1}$$

where the unknowns are E the radiative energy, $\mathbf{F}$ the radiative flux. The M_1 model is based on the entropy closure of [10] for which one can parametrize the radiative energy $E \in \mathbb{R}^+$, flux $\mathbf{F} \in \mathbb{R}^d$ and pressure $\widehat{P} \in \mathbb{R}^{d\times d}$ by

$$E = \frac{3 + \|\mathbf{u}\|^2}{3(1 - \|\mathbf{u}\|^2)^3} T^4, \qquad \mathbf{F} = \frac{-4\mathbf{u}}{3(1 - \|\mathbf{u}\|^2)^3} T^4, \qquad \widehat{P} = \left(\frac{1-\chi}{2} I + \frac{3\chi - 1}{2} \frac{\mathbf{f}\otimes\mathbf{f}}{\|\mathbf{f}\|^2}\right) E, \tag{2}$$

where $T \in \mathbb{R}^+$, $\mathbf{u} \in \mathbb{R}^d$, $\mathbf{f} = \frac{\mathbf{F}}{E}$ is the non-dimensional radiation flux, $\chi = \frac{3+4\|\mathbf{f}\|^2}{5+2\sqrt{4-3\|\mathbf{f}\|^2}}$ is the Eddington factor and $\sigma(x) > 0$ is the material opacity. Here, $\varepsilon \in]0,1[$ is a scaling parameter. For the M_1 model, the strictly concave entropy (as function of E and $\mathbf{F}$) is defined by

$$S = \frac{4}{3(1 - \|\mathbf{u}\|^2)^2} T^3 = \left(\frac{4}{3}\right) 3^{3/4} \frac{E^{3/4}(1 - \|\mathbf{u}\|^2)^{1/4}}{(3 + \|\mathbf{u}\|^2)^{3/4}} \tag{3}$$

and we have the two thermodynamics relations, see [1,4]:

$$T\,\mathrm{d}S = \mathrm{d}E - (\mathbf{u}, \mathrm{d}\mathbf{F}), \tag{4a}$$

$$q = TS - E + (\mathbf{u}, \mathbf{F}). \tag{4b}$$

In this work we are interested by the solutions of this model which satisfy the following properties:

- The maximum principle:

$$\mathscr{D} := \{E > 0,\ \|\mathbf{f}\| < 1\} = \{E > 0, \|\mathbf{F}\| < E\} \tag{5}$$

  is an invariant domain for all ε.
- The diffusion limit: at the limit when ε tends to zero, the radiative energy is solution of the diffusion equation $\partial_t E - \nabla.\frac{1}{3\sigma}\nabla E = 0$.
- The entropy inequality: $\partial_t S + \frac{1}{\varepsilon}\nabla.\mathbf{u}S \geqslant 0$ for all ε.

To our knowledge numerical methods which satisfy these requirements have been proposed in [2–4] only on 1D or 2D Cartesian meshes: this is made possible using the Jin–Levermore procedure [9] or the Gosse–Toscani method [8], which was recently adapted to unstructured meshes [6] for linear systems. The present work is a multidimensional extension of the scheme presented for the nonlinear reformulation of the M_1 model as a gas dynamic system [1,4]. The formulation as a gas dynamics system comes from the following rewriting of the M_1 model:

$$\begin{cases} \partial_t \mathbf{F} + \dfrac{1}{\varepsilon}\nabla.(\mathbf{u}\otimes\mathbf{F}) + \dfrac{1}{\varepsilon}\nabla q = -\dfrac{\sigma}{\varepsilon^2}\mathbf{F}, \\ \partial_t E + \dfrac{1}{\varepsilon}\nabla.(\mathbf{u}E + q\mathbf{u}) = 0, \end{cases} \tag{6}$$

with $q = \frac{1-\chi}{2}E$ and one finds, from (2), $\mathbf{u} = \frac{3\chi-1}{2}\frac{\mathbf{f}}{\|\mathbf{f}\|^2} = \frac{2}{3-\chi}\mathbf{f}$. In this system, the radiative pressure is splitted into a convective part and an isotropic part. The proposed numerical scheme discretizes the isotropic part with Lagrangian nodal fluxes, [7,11], and the convective part with nodal remap fluxes. We will show that the entropy inequality is sufficient to ensure the maximum principle.

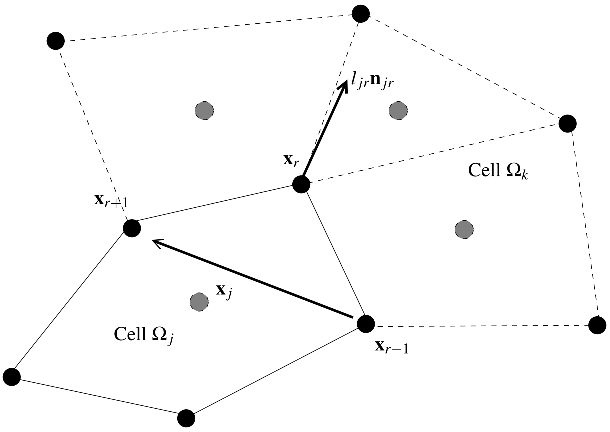

Fig. 1. Notation for the nodal formulation. Notice that $l_{jr}\mathbf{n}_{jr}$ is equal to the half of the vector that starts at $\mathbf{x}_{r-1}$ and finishes at $\mathbf{x}_{r+1}$.

2. The Eulerian scheme

Let us consider an unstructured mesh in dimension 2 (Fig. 1). The mesh is defined by a finite number of vertices $\mathbf{x}_r$ and cells Ω_j. We denote $\mathbf{x}_j$ an arbitrary point chosen inside Ω_j. For simplicity we will call this point the center of the cell. By convention the vertices are listed locally counterclockwise $\mathbf{x}_{r-1}, \mathbf{x}_r, \mathbf{x}_{r+1}$ with coordinates $\mathbf{x}_r = (x_r, y_r)$. We also define a corner length $l_{jr} = \frac{1}{2}\|\mathbf{x}_{r+1} - \mathbf{x}_{r-1}\|$ and a corner normal vector $\mathbf{n}_{jr} = \frac{1}{2l_{jr}}(-y_{r-1} + y_{r+1}, x_{r-1} - x_{r+1})^t$. To discretize the Lagrangian fluxes we use nodal schemes like [7,11] which localize the fluxes at the vertices. As in [9,6], the fluxes take into account the source term in order to give an AP scheme. We obtain the following semi-discrete scheme

$$\begin{cases} |\Omega_j|\partial_t E_j + \dfrac{1}{\varepsilon}\left(\displaystyle\sum_{R^+} l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr})E_j + \sum_{R^-} l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr})E_{k(r)}\right) + \dfrac{1}{\varepsilon}\displaystyle\sum_r (\mathbf{u}_r, \mathbf{G}_{jr}) = 0, \\ |\Omega_j|\partial_t \mathbf{F}_j + \dfrac{1}{\varepsilon}\left(\displaystyle\sum_{R^+} l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr})\mathbf{F}_j + \sum_{R^-} l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr})\mathbf{F}_{k(r)}\right) + \dfrac{1}{\varepsilon}\displaystyle\sum_r \mathbf{G}_{jr} = -\dfrac{1}{\varepsilon^2}\sum_r \sigma_r k_r \widehat{\beta}_{jr}\mathbf{u}_r, \end{cases} \tag{7}$$

with

$$\begin{cases} \mathbf{G}_{jr} = l_{jr} q_j \mathbf{n}_{jr} + c_r \widehat{\alpha}_{jr}(\mathbf{u}_j - \mathbf{u}_r) - \dfrac{\sigma_r}{\varepsilon} k_r \widehat{\beta}_{jr}\mathbf{u}_r, \\ \left(\displaystyle\sum_j c_r \widehat{\alpha}_{jr} + \dfrac{\sigma_r}{\varepsilon} k_r \sum_j \widehat{\beta}_{jr}\right)\mathbf{u}_r = \displaystyle\sum_j l_{jr} q_j \mathbf{n}_{jr} + c_r \sum_j \widehat{\alpha}_{jr}\mathbf{u}_j, \end{cases} \tag{8}$$

with the nodal quantities $\sigma_r \approx \sigma(\mathbf{x}_r)$, $k_r = \sum_j w_j \frac{(3-\chi(\mathbf{f}_j))E_j}{2}$, the speed of sound $c_r = \sum_j w_j \frac{4}{\sqrt{3}} \frac{E_j}{3+\|\mathbf{u}_j\|^2}$, $E_r = \sum_j w_j E_j$. The weights w_j are positive and satisfy $\sum_j w_j = 1$. In (7) the upwinded terms in parentheses for both equations correspond to the projection stage of a fully discrete Lagrange + projection scheme, the rest of the fluxes are equal to the Lagrange stage. This scheme is the limit as a time splitting step tends to 0 of the classical Lagrange-projection scheme for gas dynamic equations. The matrix $\widehat{\alpha}_{jr}$ is called the acoustic tensor in the context of Lagrangian fluid dynamics which can be $\widehat{\alpha}_{jr} = l_{jr}\mathbf{n}_{jr} \otimes \mathbf{n}_{jr}$, see [7], or $\widehat{\alpha}_{jr} = \frac{1}{2}(l^-_{jr}\mathbf{n}^-_{jr} \otimes \mathbf{n}^-_{jr} + l^+_{jr}\mathbf{n}^+_{jr} \otimes \mathbf{n}^+_{jr})$, see [11], with $l^{\pm}_{jr}$, $\mathbf{n}^{\pm}_{jr}$ normals and length associated to the edges $[\mathbf{x}_{r-1}, \mathbf{x}_r]$ and $[\mathbf{x}_r, \mathbf{x}_{r+1}]$. We set also $\widehat{\alpha}_r = \sum_j \widehat{\alpha}_{jr}$. The tensor $\widehat{\beta}_{jr}$ is $\widehat{\beta}_{jr} = \frac{|V_{jr}|}{|V_r|}\widehat{\beta}_r$ with $\widehat{\beta}_r = \sum_j l_{jr}\mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j)$ where V_r is the control volume and V_{jr} the part of the control volume associated to the cell j defined in [6]. The terms in (7) associated to the projection fluxes are: $R^+ = \{r/l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr}) > 0\}$, $R^- = \{r/l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr}) < 0\}$ and

$$g_{k(r)} = \frac{\sum_{j/l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr})>0} l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr}) g_j}{\sum_{j/l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr})>0} l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr})} \tag{9}$$

where $g = \mathbf{F}$ or E. As in [6], we assume that the meshes are non-degenerate that is $\widehat{\alpha}_r$ is definite positive and such that

$$(\widehat{\beta}_r \mathbf{y}, \mathbf{y}) \geqslant \beta |V_r| \, \|\mathbf{y}\|^2 \quad \beta > 0 \tag{10}$$

which is almost always the case for the meshes used, see [6] for more details.

3. Diffusion limit of the Eulerian scheme

A previous study in [6] and an asymptotic analysis based on a Hilbert expansion show that $\mathbf{u}_r$ is homogeneous to $-\frac{\nabla E}{4\sigma E}$ when ε tends to 0. Therefore, in the diffusion regime, the scheme (7) reduces to the nonlinear positive diffusion scheme with the diffusion coefficient $\frac{1}{3\sigma}$:

$$\begin{cases} \partial_t E_j(t) + \sum_r \frac{1}{12\sigma_r}\left((l_{jr}E_j\mathbf{n}_{jr} - \widehat{\beta}_{jr}\tilde{\mathbf{u}}_r), \frac{\tilde{\mathbf{u}}_r}{E_r}\right) + \sum_{R^+} \frac{1}{4\sigma_r} l_{jr}\left(\frac{\tilde{\mathbf{u}}_r}{E_r}, \mathbf{n}_{jr}\right)E_j + \sum_{R^-} \frac{1}{4\sigma_r} l_{jr}\left(\frac{\tilde{\mathbf{u}}_r}{E_r}, \mathbf{n}_{jr}\right)E_{k(r)} = 0, \\ \widehat{\beta}_r \tilde{\mathbf{u}}_r = \sum_j l_{jr}E_j\mathbf{n}_{jr}, \end{cases} \tag{11}$$

where E_{jr} and $E_{k(r)}$ are defined by the advection scheme. In [6] we prove that $\tilde{\mathbf{u}}_r$ is consistent with $-\nabla E(\mathbf{x}_r)$. The positivity of this scheme comes from (10).

4. Entropy condition and maximum principle

We first give bounds on the nodal velocities.

Proposition 1. *We assume that the computational domain is a torus. If the numerical solution lies in $\mathscr{D}$, defined by (5), then with the above assumptions on the meshes, the nodal velocities $\mathbf{u}_r$ are bounded.*

Proof. Multiplying the definition (8) for the nodal velocities by $\mathbf{u}_r$, using the above assumptions on the meshes, and the Cauchy–Schwartz inequality one obtains

$$\left(c_r\lambda_r + \frac{\sigma_r}{\varepsilon}k_r\beta|V_r|\right)\|\mathbf{u}_r\| \leqslant \sum_j l_{jr}|q_j| + c_r\sum_j l_{jr}\|\mathbf{u}_j\|$$

with λ_r the minimal eigenvalue of the symmetric and definite positive matrix $\widehat{\alpha}_r$. Using the definition of c_r and the fact that $E_j > 0$ and $||\mathbf{f}_j|| < 1$ so that $\|\mathbf{u}_j\| < 1$ we obtain that

$$\frac{E_r}{\sqrt{3}} \leqslant c_r \leqslant \frac{4E_r}{3\sqrt{3}}, \qquad 0 \leqslant q_j \leqslant \frac{1}{3}E_j, \qquad E_r \leqslant k_r \leqslant \frac{4}{3}E_r$$

and so $\|\mathbf{u}_r\| \leqslant \frac{\sum_j a_{jr}E_j}{\sum_j b_{jr}E_j}$ with positive constants a_{jr} and b_{jr} which depend on the meshes, σ and ε. Thus $\|\mathbf{u}_r\| \leqslant \max_j \frac{a_{jr}}{b_{jr}}$. This result is far from optimal but it is sufficient for our needs thereafter. □

Now we give the main result:

Lemma 2. *We assume that the computational domain is a torus. If the initial data lies in $\mathscr{D}$, defined by (5), then with the above assumptions on the meshes, the semi-discrete scheme (7)–(9) is entropic*

$$\partial_t S_j + \frac{1}{\varepsilon}\left(\sum_{R^+} l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr})S_j + \sum_{R^-} l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr})S_{k(r)}\right) \geqslant 0 \tag{12}$$

and the solution lies in $\mathscr{D}$ for all time $t \geqslant 0$. Therefore the maximum principle is preserved.

Proof. First, we remark that $\|\mathbf{F}_j(t)\| < E_j(t) \Leftrightarrow \|\mathbf{u}_j\|^2 < 1$. Using (4a) we have $\partial_t S_j = \frac{1}{T_j}(\partial_t E_j - (\mathbf{u}_j, \partial_t \mathbf{F}_j))$ and easy calculations give

$$\partial_t S_j + \frac{1}{\varepsilon}\left(\sum_{R^+} l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr})S_j + \sum_{R^-} l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr})S_{k(r)}\right) = L + P, \tag{13}$$

with

$$L = \frac{1}{T_j\varepsilon}\left(\sum_r c_r\big(l_{jr}\mathbf{n}_{jr}, (\mathbf{u}_j - \mathbf{u}_r)\big)^2 + \frac{\sigma_r}{\varepsilon}(\widehat{\beta}_r\mathbf{u}_r, \mathbf{u}_r)\right)$$

which is non-negative using the assumptions on the meshes, and

$$P = \frac{1}{T_j\varepsilon}\sum_{R^+} l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr})\big(T_jS_j - E_j + (\mathbf{u}_j, \mathbf{F}_j) - q_j\big) + \frac{1}{T_j\varepsilon}\sum_{R^-} l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr})\big(T_jS_{k(r)} - E_{k(r)} + (\mathbf{u}_j, \mathbf{F}_{k(r)}) - q_j\big).$$

Using (4b) the first term of P is equal to zero. Also with (4b), the second term of P can be recast as

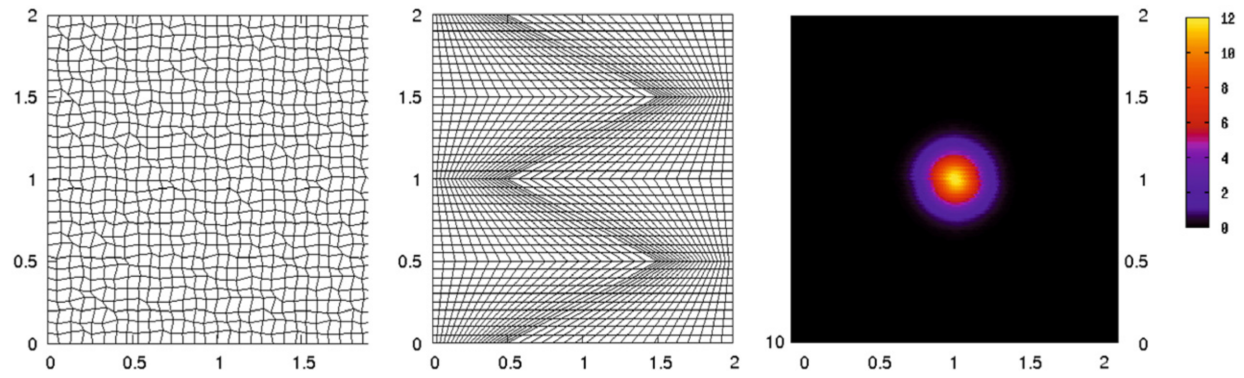

Fig. 2. Examples of meshes, random mesh and Kershaw mesh and contour plot of E for the test case for the diffusion limit regime, with $\varepsilon = 0.001$ and the Kershaw mesh.

$$\frac{1}{T_j\varepsilon}\sum_{R^-} l_{jr}(\mathbf{u}_r,\mathbf{n}_{jr})\left(S_{k(r)} - S_j - (E_{k(r)} - E_j)\frac{1}{T_j} + (\mathbf{u}_j, \mathbf{F}_{k(r)} - \mathbf{F}_j)\right) \tag{14}$$

which is positive using the concavity of the entropy. That ends the proof of (12).

Now we prove that the scheme verifies the maximum principle for all time. The semi-discrete scheme is a dynamical system $X'(t) = F(X(t))$ defined at least on $Int(\mathscr{D})$ and $F(X)$ is a piecewise C^1 function. The Cauchy–Lipschitz theorem ensures the existence of a maximal solution $X(t)$ on $[0, T_f[$.

If $E_j > 0$ and $\|\mathbf{u}_j\| < 1$ on $[0,\tau[$, $\tau \leqslant T_f$, the previous proposition shows that $\|\mathbf{u}_r\|$ is bounded. Using (12), the entropy S_j is bounded on $[0,\tau[$ from below by $S_j(t) \geqslant e^{-a_j\tau}S_j(0)$ with $a_j = \sum_{R+} l_{jr}(\mathbf{u}_r,\mathbf{n}_{jr})$. It is obvious that $\frac{E_j^{\frac{3}{4}}}{S_j} = (\frac{3}{4})3^{-3/4}\frac{(3+\|\mathbf{u}_j\|^2)^{3/4}}{(1-\|\mathbf{u}_j\|^2)^{1/4}} \geqslant \frac{3}{4}$. Using the lower bound of the entropy we obtain $E_j \geqslant (\frac{3}{4}S_j(0)e^{-a_j\tau})^{\frac{4}{3}}$. Now by the conservativity of the scheme for the E_j's and their positivity we have $E_j \leqslant \frac{\|E(0)\|_{L1}}{\min_j|\Omega_j|}$, $\|E(0)\|_{L^1} = \sum_j |\Omega_j|E_j(0)$. Using the definition of the entropy and the upper bound for E_j we obtain $(1 - \|\mathbf{u}_j\|^2) \geqslant (\frac{3}{4})^4(\frac{\min_j|\Omega_j|}{\|E(0)\|_{L^1}})^3 S(0)^4 e^{-4a_j\tau}$. Consequently on $[0,\tau[$, the quantities E_j and F_j remain in a compact strictly included in $\mathscr{D}$ i.e. the solution cannot reach $\partial\mathscr{D}$ in finite time. Thus we have proved that these quantities lie in $\mathscr{D}$ for all time $\tau < T_f$ and using classical results for dynamical system necessarily $T_f = +\infty$ otherwise the solution can be extended on some interval $[T_f, T_f + \mu[$ which is not possible since $[0, T_f]$ is the maximal interval of existence of the solution. □

5. Second order extension and numerical results

Numerical results show that the part of the limit diffusion scheme associated to the Lagrangian fluxes converge with the second order, but the part of the limit diffusion scheme associated to the advection fluxes converges only with the first order. Usually good diffusion schemes converge with the second order. Consequently we propose to add a MUSCL procedure in the advection (projection) fluxes to obtain a second order scheme in diffusive regime. The originality of the MUSCL procedure used here is in the reconstruction of the gradient of any quantities at the nodes: it provides formal second order accuracy unlike what is done in [7]. As usual any quantity f is reconstructed at the nodes as $f_{jr} = f_j - \phi(m)(\mathbf{x}_r - \mathbf{x}_j, \mathbf{g}_r)$ which is formally second order if $\phi(m) \simeq 1$ and $\mathbf{g}_r = \nabla f(\mathbf{x}_r) + O(h)$. This is true if one uses the formula $\widehat{\beta_r}\mathbf{g}_r = \sum_j l_{jr} f_j \mathbf{n}_{jr}$, see [6] for details. In our numerical tests, the limiter is defined by:

$$m = \min(m_1, m_2), \qquad m_1 = \frac{\min(f_i) - f_j}{\min(f_r) - f_j}, \qquad m_2 = \frac{\max(f_i) - f_j}{\max(f_r) - f_j},$$
$$\phi(m) = \min(\sigma, 1-\varepsilon)\max\big(0, \min(1, m)\big). \tag{15}$$

The coefficient $\min(\sigma, 1-\varepsilon)$ is introduced in the limiter so that the MUSCL procedure is effective only when ε is small that is in diffusive regime, in order to avoid breaking of the maximum principle. In practice we use this procedure to reconstruct nodal values of E an $\mathbf{F}$ which are plugged in the projection part of the scheme (7) in place of their cell centered values. Two examples of meshes used for the numerical results are given in Fig. 2.

5.1. Test case for the diffusion limit regime

To validate the limit diffusion scheme (11) we propose the following test: the initial condition is the fundamental solution of the heat equation at $t = 0.001$. The final time is $T_f = 0.011$. We compare the numerical solution of M_1 model for

Table 1
Diffusion test case: order of convergence for $\varepsilon = 0.001$ left and $\varepsilon = 0.0001$ right.

Mesh/order	80	160	320	Nb negative coef.
Cartesian mesh	1.79	1.91	1.90	0
Rand. quad mesh	1.85	1.92	1.92	0
Cartesian trig. mesh	1.90	1.95	1.92	0
Rand trig. mesh	1.92	1.95	1.92	0
Kershaw mesh	1.68	1.67	1.87	0

Mesh/order	80	160	320	Nb negative coef.
Cartesian mesh	1.79	1.94	1.98	0
Rand. quad mesh	1.85	1.94	2.	0
Cartesian trig. mesh	1.90	1.96	2.05	0
Rand trig. mesh	1.92	1.96	1.94	0
Kershaw mesh	1.69	1.67	1.89	0

Table 2
Transport test case: order of convergence.

Mesh	40–80	80–160	160–320	Nb coef. $E < 0$	Nb coef. $\|\mathbf{f}\| > 1$
Cartesian mesh	0.475	0.49	0.496	0	0
Rand. quad mesh	0.475	0.494	0.496	0	0
Kershaw mesh	0.42	0.465	0.493	0	0
Cartesian trig. mesh	0.474	0.491	0.496	0	0
Rand trig. mesh	0.477	0.492	0.496	0	0

some small values of ε and the exact solution of the diffusion equation: see Table 1 for orders of convergence and contour plot for E right picture of Fig. 2. We use a semi implicit time discretization, only the source term is treated in implicit. In Table 1 we can see that the limit diffusion scheme converges with the second order for all meshes.

5.2. *Transport test case*

To verify the respect of the maximum principle by our scheme, we propose a transport test case, $\sigma = 0$, $E(0) = F_x(0) = \mathbf{1}_{x\in[0.4:0.6]}$ and $F_y(0) = 0$. The solution is $E(t) = F_x(t) = \mathbf{1}_{x\in[0.4+t:0.6+t]}$ and $F_y(t) = 0$. The final time is $T_f = 0.2$ and we use an explicit time discretization.

Table 2 shows that the scheme preserves the maximum principle and converges, for E, with the usual order 0.5 for discontinuous solutions (for sufficiently smooth solutions order 1 is recovered), see [12] in the case of nonlinear scalar conservation laws.

6. Conclusion

In this study, we have proposed an AP scheme for the nonlinear M_1 model which preserves the maximum principle. These properties have been obtained using a reformulation of the M_1 system as gas dynamic equations. Nodal hydrodynamic schemes seems to be useful to define AP schemes on 2D distorded meshes. Moreover for the M_1 model, their entropic property gives the maximum principle. In future works, we will address implicit time discretizations and the interaction with the matter.

References

[1] C. Berthon, C. Buet, J.-F. Coulombel, B. Desprès, J. Dubois, T. Goudon, J.E. Morel, R. Turpault, Mathematical Models and Numerical Methods for Radiative Transfer, Panoramas et Synthèses, vol. 28, 2009.
[2] C. Berthon, P. Charrier, B. Dubroca, An HLLC scheme to solve the M_1 model of radiative transfer in two space dimensions, J. Sci. Comput. 31 (3) (2007) 347–389.
[3] C. Berthon, J. Dubois, B. Dubroca, T.-H. Nguyen-Bui, R. Turpault, A free streaming contact preserving scheme for the M1 model, Adv. Appl. Math. Mech. 3 (2010) 259–285.
[4] C. Buet, B. Després, Grey radiative hydrodynamics; hierarchy of models and numerical approximation, in: Mathematical Models and Numerical Methods for Radiative Transfer, in: Panor. Synthèses, vol. 28, SMF, Paris, 2009.
[5] C. Buet, B. Després, E. Franck, Asymptotic preserving finite volumes discretization for non-linear moment model on unstructured meshes, in: Proceedings FVCA6, 2011.
[6] C. Buet, B. Després, E. Franck, Design of asymptotic preserving finite volume schemes for the hyperbolic heat equation on unstructured meshes, Numer. Math. (2012), http://dx.doi.org/10.1007/s00211-012-0457-9.
[7] G. Carré, S. Del Pino, B. Després, E. Labourasse, A cell-centered Lagrangian hydrodynamics scheme on general unstructured meshes in arbitrary dimension, J. Comput. Phys. 228 (14) (2009) 5160–5518.
[8] L. Gosse, G. Toscani, An asymptotic-preserving well-balanced scheme for the hyperbolic heat equations, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 337–342.
[9] S. Jin, C.D. Levermore, Numerical schemes for hyperbolic conservation laws with stiff relaxation terms, J. Comput. Phys. 126 (1996) 449–467.
[10] C.D. Levermore, Relating Eddington factors to flux limiters, J. Quant. Spectrosc. Radiat. Transfer 31 (2) (1984) 149–160.
[11] P.-H. Maire, R. Abgrall, J. Breil, J. Ovadia, A cell-centered Lagrangian scheme for two-dimensional compressible flow problems, SIAM J. Sci. Comput. 29 (4) (2007) 1781–1824.
[12] H. Nessyahu, E. Tadmor, The convergence rate of approximate solutions for nonlinear scalar conservation laws, SIAM J. Numer. Anal. 29 (1991) 1505–1519.