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On Bellman function for extremal problems in BMO

Sur la fonction de Bellman pour des problèmes extrémaux sur l'espace BMO

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ABSTRACT

In this Note we describe our results on construction of the Bellman function solving an extremal problem for a large class of integral functionals on BMO.

RÉSUMÉ

Dans cette Note, nous décrivons nos résultats sur la construction de la fonction de Bellman qui résout un problème extrémal pour une grande classe de formes linéaires intégrales sur BMO.

1. Introduction

Symbols I and J always denote subintervals of the real line $\mathbb{R}$. We use the following notation for the average of an integrable function φ over an interval J:

$$\langle\varphi\rangle_J \stackrel{\text{def}}{=} \frac{1}{|J|}\int_J \varphi(t)\,dt.$$

Recall that the BMO space with quadratic (semi-)norm can be defined as

$$\mathrm{BMO}(I) \stackrel{\text{def}}{=} \Big\{\varphi \in L^2(I)\colon\ \|\varphi\|^2_{\mathrm{BMO}(I)} \stackrel{\text{def}}{=} \sup_{J\subset I}\big\langle|\varphi-\langle\varphi\rangle_J|^2\big\rangle_J < +\infty\Big\}.$$

Moreover, we introduce the *Bellman point* of φ:

$$\mathfrak{b}(\varphi, J) \stackrel{\text{def}}{=} \big(\langle\varphi\rangle_J, \langle\varphi^2\rangle_J\big) \in \mathbb{R}^2.$$

For $\varepsilon > 0$ fixed, consider the parabolic strip

$$\Omega_\varepsilon \stackrel{\text{def}}{=} \big\{x = (x_1, x_2) \in \mathbb{R}^2\colon\ x_1^2 \leqslant x_2 \leqslant x_1^2 + \varepsilon^2\big\}$$

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and the Bellman function

$$\mathbf{B}_\varepsilon(x; f) \stackrel{\text{def}}{=} \sup\{\langle f \circ \varphi\rangle_I :\ \|\varphi\|_{\mathrm{BMO}(I)} \leqslant \varepsilon,\ \mathfrak{b}(\varphi, I) = x\}$$

defined for $x \in \Omega_\varepsilon$.

We list some simple properties of this Bellman function:

– it does not depend on the interval I;
– it satisfies the boundary condition: $\mathbf{B}_\varepsilon(t, t^2; f) = f(t)$.

2. Our results

The main aim of the authors is to express the Bellman function $\mathbf{B}_\varepsilon(\cdot\,; f)$ in terms of f. The knowledge of an explicit formula for this function provides the sharp constants in integral inequalities on BMO. Several results of this type were achieved earlier:

– $f(t) = |t|^p$, the sharp constants in L^p estimates (see [2,5]);
– $f(t) = \chi_{(-\infty,-\lambda)\cup(\lambda,+\infty)}$, $\lambda > 0$, the weak form of the John–Nirenberg inequality (see [6,8]);
– $f(t) = e^t$, the integral form of the John–Nirenberg inequality (see [4,3]).

The next statement plays a crucial role in hunting for the function $\mathbf{B}_\varepsilon$.

Proposition 2.1. *If a locally concave function G on Ω_ε majorizes the function $\mathbf{B}_\varepsilon$ on the lower parabola, i.e., $G(t, t^2) \geqslant f(t)$, then it majorizes the function $\mathbf{B}_\varepsilon$ in the entire domain Ω_ε.*

Thus, it is reasonable to look for the minimal locally concave function B on Ω_ε that satisfies the boundary condition

$$B\big(t, t^2\big) = f(t). \tag{1}$$

In fact, the minimal locally concave function coincides with the required function $\mathbf{B}_\varepsilon$. Indeed, clearly, it dominates $\mathbf{B}_\varepsilon$. In order to prove the converse inequality for every point $x \in \Omega_\varepsilon$, it suffices to find a function $\varphi \in \mathrm{BMO}_\varepsilon(I)$ such that $x = \mathfrak{b}(\varphi, I)$ and $B(x) = \langle f(\varphi)\rangle_I$. These functions are called optimizers.

Theorem 2.2. *Suppose $\varepsilon_0 > 0$, the function f lies in $C^2(\mathbb{R}) \cap W_1^3(\mathbb{R}, e^{-|t|/\varepsilon_0})$, and the following additional conditions on the third derivative are satisfied: $f''' \neq 0$ a.e. on $\mathbb{R}$, there are only finitely many points where f''' changes its sign, and these points are pairwise separated at least by $2\varepsilon_0$. Then the Bellman function $\mathbf{B}_\varepsilon$ coincides with the minimal locally concave function on Ω_ε that satisfies the boundary condition (1). Moreover, $\mathbf{B}_\varepsilon \in C^1(\Omega_\varepsilon)$.*

Any smooth locally concave function B has negative semidefinite Hessian. What is more, such a minimal function satisfies the homogeneous Monge–Ampère equation

$$\det(H) = 0; \qquad H = \begin{pmatrix} \frac{\partial^2 B}{\partial x_1^2} & \frac{\partial^2 B}{\partial x_1 \partial x_2} \\ \frac{\partial^2 B}{\partial x_1 \partial x_2} & \frac{\partial^2 B}{\partial x_2^2} \end{pmatrix}. \tag{2}$$

The general theory of Monge–Ampère equations (see, for example, [7]) asserts that the integral curves (extremals) of the vector field generated by the kernel of the Hessian are line segments. First derivatives of B are constant along these extremals and the function B itself is linear on them. We find the geometric picture of the foliation of Ω_ε by the extremals and by subdomains where the Hessian is zero. If the foliation is found, one can recover the function B by using its linearity on the extremals and minimality.

Taking advantage of minimality, we can assert that the extremals cannot cross the upper bound of Ω_ε transversally and cannot cross each other. Moreover, either both endpoints of each extremal lie on the lower parabola or one lies on the lower parabola and the other on the upper parabola. In the second case, the extremal must touch the upper parabola (see Fig. 2). In addition, the subdomains where B is linear can occur. If the extremal connects two points on the lower parabola, then one can easily restore the function B on this extremal by using linearity and the boundary condition (1). If the extremal touches the upper parabola, then we can choose the slope coefficient. To be precise, the function B on such an extremal can be defined as

$$B(x_1, x_2) = f(u) + m(u)(x_1 - u), \tag{3}$$

where u is the abscissa of the endpoint that lies on the lower parabola, and $m = m(u)$ is the slope coefficient on this extremal.

The next statement helps to find the foliation and the function m for f given.

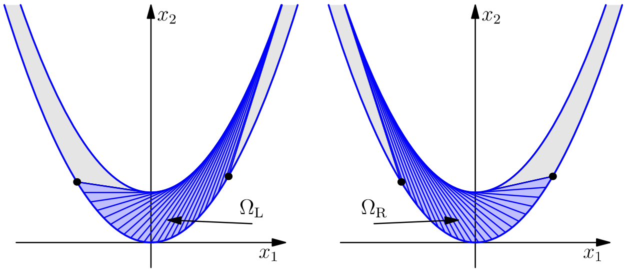

Fig. 1. The domains Ω_L and Ω_R.

Fig. 1. Les domaines Ω_L et Ω_R.

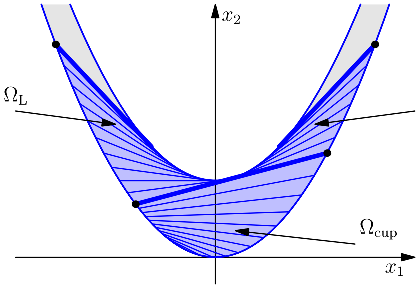

Fig. 2. A cup Ω_{cup} lying between Ω_L and Ω_R.

Fig. 2. Une écuelle Ω_{cup} entre Ω_L et Ω_R.

Proposition 2.3. *Let B be defined by* (3) *in a subdomain Ω_R (Ω_L) that is foliated by the right* (*left*) *tangents to the upper parabola* (*see Fig.* 1) *and satisfies the boundary condition* (1)*. Then its first derivatives $\frac{\partial B}{\partial x_1}$, $\frac{\partial B}{\partial x_2}$ are constant on the extremals if and only if*

$$\pm\varepsilon m'(u) + m(u) = f'(u) \tag{4}$$

(*here + corresponds to the right tangents and − to the left ones*)*. The function B is locally concave if and only if $\pm m''(u) \leqslant 0$.*

The next statement treats the subdomain Ω_{cup} (see Fig. 2) foliated by the extremals with the two endpoints on the lower bound.

Proposition 2.4. *Let B be linear on extremals in Ω_{cup} and let the boundary condition* (1) *be satisfied for it. Then its first derivatives $\frac{\partial B}{\partial x_1}$, $\frac{\partial B}{\partial x_2}$ are constant along the extremals if and only if*

$$\frac{f'(a) + f'(b)}{2} = \langle f' \rangle_{[a,b]},$$

where (a, a^2) and (b, b^2) are the endpoints of the extremals. The concavity of B can also be rewritten as differential inequalities: $f''(a) \leqslant \langle f'' \rangle_{[a,b]}$ and $f''(b) \leqslant \langle f'' \rangle_{[a,b]}$.

It turns out that subdomains foliated by such extremals can be found near the points where f''' changes its sign from + to − only. We call such subdomains *cups*.

In the case when a subdomain of linearity borders two subdomains of tangents oriented in a different way, an *angle* arises (see Fig. 3). The continuity and the concavity of the glued function are equivalent to the following equation:

$$\lim_{u \to v+} m''(u) + \lim_{u \to v-} m''(u) = 0,$$

where the point (v, v^2) is the vertex of the angle.

The same equations arise when a subdomain of linearity glues with a cup and two subdomains foliated by tangents of the same direction (see Fig. 4). We call such construction *trolleybuses*.

The next theorem was proved in the paper [1].

Theorem 2.5. *Under the assumptions of Theorem 2.2, there exists a collection of cups, angles, trolleybuses, and subdomains foliated by tangents such that all the corresponding equations and inequalities are fulfilled.*

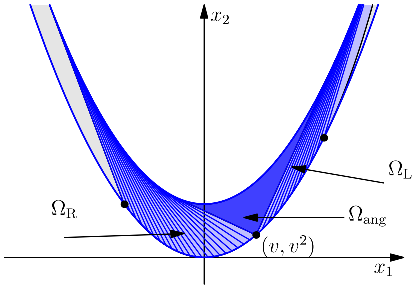

Fig. 3. An angle $\Omega_{\rm ang}$ lying between $\Omega_{\rm R}$ and $\Omega_{\rm L}$.

Fig. 3. Un angle $\Omega_{\rm ang}$ entre $\Omega_{\rm R}$ et $\Omega_{\rm L}$.

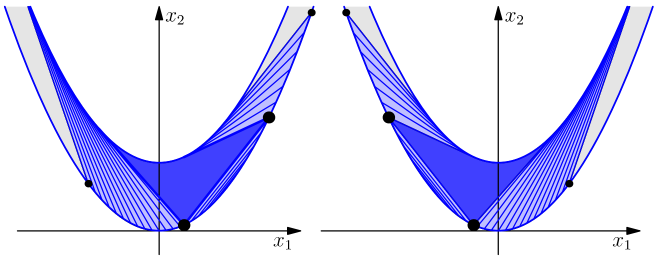

Fig. 4. A right $\Omega_{\rm tr,R}$ and a left $\Omega_{\rm tr,L}$ trolleybuses.

Fig. 4. Un trolleybus droit $\Omega_{\rm tr,R}$ et un trolleybus gauche $\Omega_{\rm tr,L}$.

A short algorithm for finding such a collection for ε fixed is provided as the proof of that theorem. The function B can be recovered by the foliation easily. Inside the cups it can be found by using linearity on the extremals. In the subdomains of tangents one needs to solve the differential equations (4) on the function m, and B can be recovered by (3). In the subdomains of linearity (trolleybuses or angles) B can be recovered by linearity.

Details of these constructions and proofs can be found in [1].

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References

[1] P. Ivanishvili, N. Osipov, D. Stolyarov, V. Vasyunin, P. Zatitskiy, Bellman function for extremal problems in BMO, preprint, http://arxiv.org/abs/1205.7018, 2012.

[2] L. Slavin, V. Vasyunin, Sharp L^p estimates on BMO, preprint, http://arxiv.org/abs/1110.1771, 2011.

[3] L. Slavin, V. Vasyunin, Sharp results in the integral-form John–Nirenberg inequality, Trans. Amer. Math. Soc. 363 (8) (2011) 4135–4169; preprint, http://arxiv.org/abs/0709.4332, 2007.

[4] V. Vasyunin, The sharp constant in the John–Nirenberg inequality, preprint POMI, No. 20, http://www.pdmi.ras.ru/preprint/2003/index.html, 2003.

[5] V.I. Vasyunin, Mutual estimates of L^p-norms, and the Bellman function, Zapiski POMI 355 (2008) 81–138 (in Russian); English translation in: J. Math. Sci. 156 (5) (2009) 766–798.

[6] V. Vasyunin, Sharp constants in the classical weak form of the John–Nirenberg inequality, preprint POMI, No. 10, http://www.pdmi.ras.ru/preprint/2011/eng-2011.html, 2011.

[7] Vasily Vasyunin, Alexander Volberg, Monge–Ampère equation and Bellman optimization of Carleson embedding theorems, Amer. Math. Soc. Transl. Ser. 2 226 (2009) 195–238.

[8] Vasily Vasyunin, Alexander Volberg, Sharp constants in the classical weak form of the John–Nirenberg inequality, preprint, http://arxiv.org/abs/1204.1782, 2012.