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Regularity criteria for weak solutions to the Navier–Stokes equations based on spectral projections of vorticity

Critères de régularité des solutions faibles des équations de Navier–Stokes fondés sur les projections spectrales de la vorticité

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ABSTRACT

We denote $P_a^+ := \int_a^\infty \mathrm{d}E_\lambda$, where $\{E_\lambda\}$ is the spectral resolution of identity associated with the self-adjoint operator **curl** in the space $\mathbf{L}^2_\sigma(\mathbb{R}^3)$. Further, we denote $\boldsymbol{\omega}_a^+ := P_a^+ \, \mathbf{curl}\, \mathbf{v}$, where $\mathbf{v}$ is a weak solution to the Navier–Stokes initial value problem in $\mathbb{R}^3 \times (0,T)$. We assume that $a = a(t)$ is a real function in $(0,T)$. We show that certain conditions imposed on function a and $\boldsymbol{\omega}_a^+$, or only on the third component ω_{a3}^+ of $\boldsymbol{\omega}_a^+$, imply regularity of solution $\mathbf{v}$.

RÉSUMÉ

Soit $\{E_\lambda\}$ la résolution spectrale de l'identité associée à l'opérateur auto-adjoint **curl** dans l'espace $\mathbf{L}^2_\sigma(\mathbb{R}^3)$, et soit $a = a(t)$ une fonction à valeurs réelles définie sur $(0,T)$. On note $P_a^+ := \int_a^\infty \mathrm{d}E_\lambda$, puis, lorsque $\mathbf{v}$ est solution faible du problème de condition initiale de Navier–Stokes dans $\mathbb{R}^3 \times (0,T)$, $\boldsymbol{\omega}_a^+ := P_a^+ \, \mathbf{curl}\, \mathbf{v}$. On établit alors la régularité de $\mathbf{v}$ sous certaines conditions imposées à a et, ou bien à $\boldsymbol{\omega}_a^+$, ou bien à sa troisième composante ω_{a3}^+ seulement.

Version française abrégée

Dans cette Note, nous considérons les solutions faibles du système des équations de Navier–Stokes décrivant en espace et en temps le champ de vitesse $\mathbf{v}$ (et le champ de pression p associé) d'un écoulement incompressible dans l'espace tridimensionnel entier. Même si la donnée initiale est plus régulière que nécessaire pour assurer leur existence, la régularité des solutions faibles est toujours un sujet de recherche. De nombreuses tentatives ont mis en évidence quelles propriétés supplémentaires d'intégrabilité pour une solution faible $\mathbf{v}$ permettaient de contrôler sa régularité. Pour une vue générale de cette approche par critères a posteriori, on pourra consulter, par exemple, [5] ou [10]. Nous semblent particulièrement intéressants les critères a posteriori formulés sur une composante seulement du champ de vitesse $\mathbf{v}$ ou sur quelques composantes seulement de son gradient $\nabla\mathbf{v}$ ou de la vorticité $\boldsymbol{\omega} = \mathbf{curl}\,\mathbf{v}$. La question est ouverte de savoir si la régularité d'une solution faible $\mathbf{v}$ peut être contrôlée par une composante seulement de la vorticité. La présente Note apporte une

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contribution dans cette direction, en supposant des propriétés supplémentaires d'intégrabilité pour une ou plusieurs composantes de la projection spectrale de la vorticité.

Posons $A := |\,\mathbf{curl}\,| = \int_{-\infty}^{+\infty} |\lambda| \, \mathrm{d}E_\lambda$, avec les notations de projections spectrales introduites dans le résumé. A est un opérateur positif, auto-adjoint dans $\mathbf{L}^2_\sigma(\mathbb{R}^3)$, et comme on peut le vérifier $A = S^{1/2}$ où $S := \mathbf{curl}^2$ est l'opérateur de Stokes. Les opérateurs S et A satisfont l'inégalité de type Sobolev $\|\mathbf{u}\|_{3;\mathbb{R}^3} \leqslant c_1 \|A^{1/2}\mathbf{u}\|_{2;\mathbb{R}^3} = c_1 \|S^{1/4}\mathbf{u}\|_{2;\mathbb{R}^3}$ pour $\mathbf{u} \in D(A^{1/2})$. Le théorème suivant énonce nos critères de régularité (dans le cas où la fonction a est identiquement nulle, sa généralisation dans le cas contraire est énoncée dans la partie anglaise) :

Théorème. *Soit* $\mathbf{v}$ *une solution faible du problème de Cauchy de condition initiale* $\mathbf{v}_0 \in \mathbf{L}^2_\sigma(\mathbb{R}^3)$ *pour les équations de Navier–Stokes. On suppose satisfaites au moins une des deux conditions,*

(i) $(-\Delta)^{1/4}\boldsymbol{\omega}^+ \in \mathbf{L}^2(\mathbb{R}^3 \times (0,T))$,
(ii) $(-\Delta)^{3/4}\omega_3^+ \in L^2(\mathbb{R}^3 \times (0,T))$,

et au moins une des deux conditions,

(a) $\mathbf{v}_0 \in \mathbf{L}^2_\sigma(\mathbb{R}^3)$ *et* $\mathbf{v}$ *vérifie l'inégalité d'énergie dans sa formulation forte, ou*
(b) $\mathbf{v}_0 \in D(A^{1/2})$ *et* $\mathbf{v}$ *vérifie l'inégalité d'énergie.*

Alors $\mathbf{v}$ *ne présente aucun point singulier dans* $\mathbb{R}^3 \times (0,T)$*, et la norme* $\|A^{1/2}\mathbf{v}\|_{2;\mathbb{R}^3}$ *est bornée dans chaque intervalle* (ϑ, T)*, où* $0 < \vartheta < T$*. En outre,* $\|A^{1/2}\mathbf{v}\|_{2;\mathbb{R}^3}$ *est bornée dans tout l'intervalle* $(0,T)$ *si la condition* (b) *est satisfaite.*

La structure, connue depuis J. Leray, des solutions faibles $\mathbf{v}$ vérifiant l'inégalité d'énergie dans sa formulation forte, joue un premier rôle essentiel dans la démonstration. Il existe sur l'axe temporel une famille d'intervalles ouverts disjoints $\{(a_\gamma, b_\gamma)\}_{\gamma\in\Gamma}$ et un ensemble de mesure nulle M tels que $(0,T) = \bigcup_{\gamma\in\Gamma}(a_\gamma, b_\gamma) \cup M$ avec les propriétés : (a1) $\mathbf{v}|_{\mathbb{R}^3\times(a_\gamma,b_\gamma)}$ est de classe C^∞ quelque soit γ, (a2) $\|A^{1/2}\mathbf{v}\|_{2;\mathbb{R}^3}$ est borné dans chaque intervalle (a_γ, b_γ), et (a3) $\mathbf{v}$ peut éventuellement présenter des points $(\mathbf{x}, t)$ singuliers aux instants $t = b_\gamma$ au sens où $\|A^{1/2}\mathbf{v}(t)\|_{2;\mathbb{R}^3} \to \infty$ lorsque $t \to b_\gamma -$.

Soit $\mathbf{v}$ une telle solution faible. Soit $\gamma \in \Gamma$ fixé et $\tau \in (a_\gamma, b_\gamma)$. En utilisant l'inégalité de type Sobolev rappelée ci-dessus, l'inéquation suivante est assez immédiate à obtenir pour $t \in [\tau, b_\gamma)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|A^{1/2}\mathbf{v}(t)\big\|^2_{2;\mathbb{R}^3} + \big\|A^{3/2}\mathbf{v}(t)\big\|^2_{2;\mathbb{R}^3} \leqslant 4c_1^6 \big\|A^{1/2}\mathbf{v}(t)\big\|^2_{2;\mathbb{R}^3}\big\|A^{1/2}\boldsymbol{\omega}^+(t)\big\|^2_{2;\mathbb{R}^3},$$

où $\boldsymbol{\omega}^+ = \int_0^\infty \mathrm{d}E_\lambda(\boldsymbol{\omega}) = \int_0^\infty \lambda \, \mathrm{d}E_\lambda(\mathbf{v})$ n'est autre que la vorticité de $\mathbf{v}^+ = \int_0^\infty \mathrm{d}E_\lambda(\mathbf{v})$.

La structure de $(-\Delta)^{1/4}\boldsymbol{\omega}^+$, que nous proposons d'étudier, joue le second rôle essentiel, pour estimer $\|A^{1/2}\boldsymbol{\omega}^+(t)\|^2_{2;\mathbb{R}^3} = (A\boldsymbol{\omega}^+(t), \boldsymbol{\omega}^+(t))_{2;\mathbb{R}^3} = (\mathbf{curl}\,\boldsymbol{\omega}^+(t), \boldsymbol{\omega}^+(t))_{2;\mathbb{R}^3}$, et pour justifier l'inégalité

$$\big\|A^{1/2}\boldsymbol{\omega}^+(t)\big\|^2_{2;\mathbb{R}^3} \leqslant \big\|\boldsymbol{\omega}^+(t)\big\|^2_{2;\mathbb{R}^3} + \big\|(-\Delta)^{3/4}\boldsymbol{\omega}_3^+(t)\big\|^2_{2;\mathbb{R}^3},$$

auquel cas le lemme de Gronwall permet de conclure et d'exclure b_γ comme instant de singularité. Nous donnons davantage d'explications dans la partie anglaise, un procédé standard de localisation dans des cylindres parallèles au troisième axe recouvrant $\mathbb{R}^3$ et l'introduction de deux fonctions y et z définies à l'intersection de deux cylindres « voisins » jouent le troisième rôle essentiel pour la démonstration.

1. Introduction and formulation of the main results

Let $T > 0$. We denote $Q_T := \mathbb{R}^3 \times (0,T)$. We deal with the Navier–Stokes initial value problem

$$\partial_t\mathbf{v} + \boldsymbol{\omega} \times \mathbf{v} + \mathbf{curl}^2\,\mathbf{v} = -\nabla\left(p + \frac{1}{2}|\mathbf{v}|^2\right) \quad \text{in } Q_T, \tag{1}$$

$$\operatorname{div}\mathbf{v} = 0 \quad \text{in } Q_T, \tag{2}$$

$$\mathbf{v}(\mathbf{x}, .) \to 0 \quad \text{for } |\mathbf{x}| \to \infty, \tag{3}$$

$$\mathbf{v}(\,.\,,0) = \mathbf{v}_0 \quad \text{in } \mathbb{R}^3 \tag{4}$$

for the unknown velocity $\mathbf{v} = (v_1, v_2, v_3)$ and pressure p. We denote by $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) := \mathbf{curl}\,\mathbf{v}$ the vorticity of flow $\mathbf{v}$.

We assume that $\mathbf{v}$ is a weak solution of the problem (1)–(4). Weak solutions of the problem (1)–(4) in the space $L^2(0,T;\mathbf{W}^{1,2}(\mathbb{R}^3)) \cap L^\infty(0,T;\mathbf{L}^2(\mathbb{R}^3))$ exist, provided that the initial velocity $\mathbf{v}_0 \in \mathbf{L}^2(\mathbb{R}^3)$ is divergence-free in the sense of distributions. A *regular point* of solution $\mathbf{v}$ is defined as a point $(\mathbf{x}, t) \in Q_T$ such that there exists a space–time neighborhood U of $(\mathbf{x}, t)$ so that $\mathbf{v}$ is essentially bounded in U. Points in Q_T that are not regular, are called *singular.* The question whether a singular point can develop in a weak solution $\mathbf{v}$ is a crucial open problem in the theory of the Navier–Stokes equations.

There exist many a posteriori criteria, saying that if a weak solution has some additional properties then it has no singular points in Q_T or in a sub-domain $D \subset Q_T$. Exact citations and further details on this topic can be found in the survey papers [5] and [10]. Some criteria impose conditions only on some components of velocity $\mathbf{v}$ or its gradient $\nabla\mathbf{v}$ or the corresponding vorticity $\boldsymbol{\omega}$: the problem is considered in a domain $\Omega \subset \mathbb{R}^3$ and the component v_3 is assumed to be essentially bounded in a space–time region $D \subset \Omega \times (0,T)$ in [8]. Then $\mathbf{v}$ has no singular points in D. This result has been later several times improved in [9,10,7,2], and finally [15], where the authors assume that $v_3 \in L^r(0,T;L^s(\mathbb{R}^3))$ with $2/r + 3/s \leqslant \frac{3}{4} + 1/(2s)$, $s > \frac{10}{3}$. Of a series of papers, where the regularity of weak solution $\mathbf{v}$ is studied in dependence on certain integrability properties of some components of the tensor $\nabla\mathbf{v}$, we mention [1,3,6,7,15,12,13]. In paper [3], the authors prove regularity of solution $\mathbf{v}$ by means of conditions imposed on only two components of vorticity. They assume that the initial velocity $\mathbf{v}_0$ is "smooth" and $\omega_1, \omega_2 \in L^r(0,T;L^s(\mathbb{R}^3))$ with $1 < r < \infty$, $\frac{3}{2} < s < \infty$, $2/r + 3/s \leqslant 2$, or the norms of ω_1 and ω_2 in $L^\infty(0,T;L^{3/2}(\mathbb{R}^3))$ are "sufficiently small". The cited criteria that concern the interior regularity hold for the so-called suitable weak solution, because here one needs to apply an appropriate localization procedure (see e.g. [10]).

The question, whether the regularity of weak solution $\mathbf{v}$ can be controlled by only one component of vorticity, is open.

The a posteriori criteria can be considered to be attempts to find a minimum quantity which controls the regularity of the weak solution (i.e. whose smoothness or certain rate of integrability implies that the whole solution is smooth). The presented Note brings a contribution to this field. The quantity which is assumed to be "smooth" in this Note is either a certain spectral projection of vorticity or only one component of this spectral projection.

1.1. Notation and auxiliary results

The norm in $L^q(\mathbb{R}^3)$ (or in $\mathbf{L}^q(\mathbb{R}^3)$, which is the space of vector functions) is denoted by $\|\,.\,\|_{q;\mathbb{R}^3}$. The scalar product in $\mathbf{L}^2(\mathbb{R}^3)$ is denoted by $(.,.)_{2;\mathbb{R}^3}$. The norm in $W^{s,q}(\mathbb{R}^3)$ (or $\mathbf{W}^{s,q}(\mathbb{R}^3)$) is denoted by $\|\,.\,\|_{s,q;\mathbb{R}^3}$. Other norms and scalar products are denoted by analogy. The completion of $\mathbf{C}^\infty_{0,\sigma}(\mathbb{R}^3)$ (the linear space of all infinitely differentiable divergence-free vector functions in $\mathbb{R}^3$, with a compact support in $\mathbb{R}^3$) in $\mathbf{L}^2(\mathbb{R}^3)$ is denoted by $\mathbf{L}^2_\sigma(\mathbb{R}^3)$. The intersection $\mathbf{W}^{1,2}(\mathbb{R}^3) \cap \mathbf{L}^2_\sigma(\mathbb{R}^3)$ is denoted by $\mathbf{W}^{1,2}_\sigma(\mathbb{R}^3)$.

The Stokes operator $S := \mathbf{curl}^2$, as an operator in space $\mathbf{L}^2_\sigma(\mathbb{R}^3)$, coincides with the reduction of $(-\Delta)$ to $\mathbf{L}^2_\sigma(\mathbb{R}^3)$, see e.g. [14, p. 138]. The domain of S is the space $\mathbf{W}^{2,2}(\mathbb{R}^3) \cap \mathbf{L}^2_\sigma(\mathbb{R}^3)$. Operator S is positive, its spectrum is continuous and covers the interval $[0,\infty)$, see [4]. The power $S^{1/4}$ of operator S satisfies the Sobolev-type inequality $\|\mathbf{u}\|_{3;\mathbb{R}^3} \leqslant c_1 \|S^{1/4}\mathbf{u}\|_{2;\mathbb{R}^3}$ for $\mathbf{u} \in D(S^{1/4})$, see [14, p. 141].

Operator $\mathbf{curl}$, with the domain $D(\mathbf{curl}) := \mathbf{W}^{1,2}_\sigma(\mathbb{R}^3)$, is self-adjoint in $\mathbf{L}^2_\sigma(\mathbb{R}^3)$. The spectrum of $\mathbf{curl}$ is continuous and coincides with the whole real axis, see [11]. We denote by $\{E_\lambda\}$ the spectral resolution of identity, associated with operator $\mathbf{curl}$. Projection E_λ is strongly continuous in dependence on λ, because the spectrum of $\mathbf{curl}$ consists only of the continuous part. We denote $P^- := E_0 = \int_{-\infty}^0 \mathrm{d}E_\lambda$ and $P^+ := I - E_0 = \int_0^\infty \mathrm{d}E_\lambda$. Operators P^- and P^+ are orthogonal and complementary projections in $\mathbf{L}^2_\sigma(\mathbb{R}^3)$. We put $A := |\,\mathbf{curl}\,| = \int_{-\infty}^\infty |\lambda|\,\mathrm{d}E_\lambda$. Operator A is positive, self-adjoint, and $A = S^{1/2}$. (The last identity follows from the facts that $F_\lambda := O$ for $\lambda < 0$, $F_\lambda := E_\lambda - E_{-\lambda}$ for $\lambda \geqslant 0$, is the spectral resolution of identity associated with operator A, and $G_\lambda := O$ for $\lambda < 0$, $G_\lambda := F_{\sqrt{\lambda}}$ for $\lambda \geqslant 0$, is the resolution of identity associated with operator S.)

We denote $\mathbf{v}^- := P^-\mathbf{v}$, $\mathbf{v}^+ := P^+\mathbf{v}$, $\boldsymbol{\omega}^- := P^-\boldsymbol{\omega}$ and $\boldsymbol{\omega}^+ := P^+\boldsymbol{\omega}$. The components of $\mathbf{v}^+$ are denoted by v_1^+, v_2^+, v_3^+, the components of functions $\mathbf{v}^-$, $\boldsymbol{\omega}^-$ and $\boldsymbol{\omega}^+$ are denoted by analogy. Operator $\mathbf{curl}$ commutes with projections P^- and P^+, hence $\boldsymbol{\omega}^- = \mathbf{curl}\,\mathbf{v}^- = -A\mathbf{v}^-$ and $\boldsymbol{\omega}^+ = \mathbf{curl}\,\mathbf{v}^+ = A\mathbf{v}^+$. Function $\mathbf{v}^+$ can be heuristically understood to be the part of velocity that consists of infinitely many "infinitely small" contributions $\mathrm{d}E_\lambda(\mathbf{v})$. The contributions are summed over $\lambda > 0$ and each of them is a "positive" Beltrami flow, i.e. a flow whose vorticity is a positive multiple of velocity. (Here, concretely, $\mathbf{curl}\,\mathrm{d}E_\lambda(\mathbf{v}) = \lambda\,\mathrm{d}E_\lambda(\mathbf{v})$.)

We denote by (EI) the energy inequality and by (SEI) the so-called strong energy inequality. The forms and meanings of both the inequalities are explained e.g. in [5]. The existence of weak solutions satisfying both (EI) and (SEI) is well known, see [5].

1.2. The main results

The next two theorems bring the main results of this paper. Theorem 2 is a generalization of Theorem 1. Both the theorems show that an eventual singularity of solution $\mathbf{v}$ must at the same time develop in the "positive part" $\mathbf{v}^+$ as well as in the "negative part" $\mathbf{v}^-$. Moreover, it is seen from Theorem 2 that the singularity is especially a matter of behavior of the part of $\mathbf{v}^+$ (or $\mathbf{v}^-$) which corresponds to high-frequency Beltrami flows.

Theorem 1. *Suppose that $\mathbf{v}$ is a weak solution to the problem (1)–(4), such that either* (a) $\mathbf{v}_0 \in \mathbf{L}^2_\sigma(\mathbb{R}^3)$ *and $\mathbf{v}$ satisfies* (SEI), *or* (b) $\mathbf{v}_0 \in D(A^{1/2})$ *and $\mathbf{v}$ satisfies* (EI). *Assume that at least one of the two conditions*

(i) $(-\Delta)^{1/4}\boldsymbol{\omega}^+ \in \mathbf{L}^2(Q_T)$,
(ii) $(-\Delta)^{3/4}\omega_3^+ \in L^2(Q_T)$

holds. Then the norm $\|A^{1/2}\mathbf{v}\|_{2;\mathbb{R}^3}$ *is bounded in each time interval* (ϑ, T) *where* $0 < \vartheta < T$*. Consequently, solution* $\mathbf{v}$ *has no singular points in* Q_T*. If condition* (b) *holds then* $\|A^{1/2}\mathbf{v}\|_{2;\mathbb{R}^3}$ *is bounded on the whole interval* $(0, T)$*.*

Suppose that $a = a(t)$ is a real function in the interval $(0, T)$. We denote $P^+_{a(t)} := I - E_{a(t)} = \int_{a(t)}^{\infty} \mathrm{d}E_\lambda$, $\mathbf{v}^+_a(t) := P^+_{a(t)}\mathbf{v}(t)$, and $\boldsymbol{\omega}^+_a(t) := P^+_{a(t)}\boldsymbol{\omega}(t) = \mathbf{curl}\,\mathbf{v}^+_a(t)$. The third component of function $\boldsymbol{\omega}^+_a$ is denoted by ω^+_{a3}. Further, we denote by a^+ the positive part of function a and by a^- the negative part of a.

Theorem 2. *Let the assumptions of Theorem 1 be fulfilled, with the conditions*

(i)′ $a^+ \in L^3(0, T)$ *and* $(-\Delta)^{1/4}\boldsymbol{\omega}^+_a \in \mathbf{L}^2(Q_T)$,
(ii)′ $a^+ \in L^3(0, T)$, $a^- \in L^5(0, T)$ *and* $(-\Delta)^{3/4}\omega^+_{a3} \in L^2(Q_T)$

instead of (i) *and* (ii)*. Then the conclusions of Theorem 1 are true.*

2. On the proofs of Theorems 1 and 2

We only sketch the proofs of Theorems 1 and 2 in this section. All the details can be found in [11].

Multiplying Eq. (1) by $A\mathbf{v}$, using the identities $\boldsymbol{\omega} = \boldsymbol{\omega}^- + \boldsymbol{\omega}^+$, $A\mathbf{v} = -\boldsymbol{\omega}^- + \boldsymbol{\omega}^+$, and integrating in $\mathbb{R}^3$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\big\|A^{1/2}\mathbf{v}\big\|^2_{2;\mathbb{R}^3} - 2\big(\boldsymbol{\omega}^+ \times \mathbf{v}, \boldsymbol{\omega}^-\big)_{2;\mathbb{R}^3} + \big\|A^{3/2}\mathbf{v}\big\|^2_{2;\mathbb{R}^3} = 0. \tag{5}$$

Using the inequality $\|\mathbf{v}\|_{3;\mathbb{R}^3} \leqslant c_1\|A^{1/2}\mathbf{v}\|_{2;\mathbb{R}^3}$, we can estimate the term $(\boldsymbol{\omega}^+ \times \mathbf{v}, \boldsymbol{\omega}^-)_{2;\mathbb{R}^3}$ as follows:

$$\begin{aligned}\big|\big(\boldsymbol{\omega}^+ \times \mathbf{v}, \boldsymbol{\omega}^-\big)_{2;\mathbb{R}^3}\big| &\leqslant \big\|\boldsymbol{\omega}^+\big\|_{3;\mathbb{R}^3}\|\mathbf{v}\|_{3;\mathbb{R}^3}\big\|\boldsymbol{\omega}^-\big\|_{3;\mathbb{R}^3} \leqslant c_1^3\big\|A^{1/2}\boldsymbol{\omega}^+\big\|_{2;\mathbb{R}^3}\big\|A^{1/2}\mathbf{v}\big\|_{2;\mathbb{R}^3}\big\|A^{1/2}\boldsymbol{\omega}^-\big\|_{2;\mathbb{R}^3} \\ &\leqslant \frac{1}{4}\big\|A^{3/2}\mathbf{v}\big\|^2_{2;\mathbb{R}^3} + c_1^6\big\|A^{1/2}\mathbf{v}\big\|^2_{2;\mathbb{R}^3}\big\|A^{1/2}\boldsymbol{\omega}^+\big\|^2_{2;\mathbb{R}^3}.\end{aligned}$$

Thus, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|A^{1/2}\mathbf{v}\big\|^2_{2;\mathbb{R}^3} + \big\|A^{3/2}\mathbf{v}\big\|^2_{2;\mathbb{R}^3} \leqslant 4c_1^6\big\|A^{1/2}\mathbf{v}\big\|^2_{2;\mathbb{R}^3}\big\|A^{1/2}\boldsymbol{\omega}^+\big\|^2_{2;\mathbb{R}^3}. \tag{6}$$

If condition (i) of Theorem 1 is fulfilled then the term $\|A^{1/2}\boldsymbol{\omega}^+\|^2_{2;\mathbb{R}^3}$ on the right hand side of (6) is in $L^1(0, T)$. Thus, we can apply Gronwall's inequality to (6) and show that $\|A^{1/2}\mathbf{v}\|_{2;\mathbb{R}^3}$ is bounded on the interval $[\tau, T)$ for each $\tau \in (0, T)$.

Note that the so-called helicity $H(\mathbf{v}) := (\mathbf{v}, \mathbf{curl}\,\mathbf{v})_{2;\mathbb{R}^3}$ equals $H(\mathbf{v}^+) + H(\mathbf{v}^-)$ (the first term being non-negative and the second term being non-positive), while $\|A^{1/2}\mathbf{v}\|^2_{2;\mathbb{R}^3} = H(\mathbf{v}^+) - H(\mathbf{v}^-)$. Thus, condition (i) implies that both the terms $H(\mathbf{v}^+)$ and $H(\mathbf{v}^-)$ are in $L^\infty(\tau, T)$ for each $\tau \in (0, T)$.

Let us further assume that condition (ii) of Theorem 1 holds. This case is much more subtle. The crucial part is the estimate of $\|A^{1/2}\boldsymbol{\omega}^+\|^2_{2;\mathbb{R}^3}$. In order to reduce the problem to domains where we can apply Poincaré-type inequalities, we construct a partition $\{\boldsymbol{\omega}^{mn}\}_{m,n\in\mathbb{Z}}$ of function $\boldsymbol{\omega}^+$.

For $m, n \in \mathbb{Z}$ and $\xi \in (-\frac{1}{2}, \infty)$, we denote $K^{mn}_\xi := (m - \xi, m + 1 + \xi) \times (n - \xi, n + 1 + \xi) \subset \mathbb{R}^2$. Further, we put $C^{mn} := K^{mn}_2 \times \mathbb{R} = (m - 2, m + 3) \times (n - 2, n + 3) \times \mathbb{R} \subset \mathbb{R}^3$. Let $\epsilon \in (0, \frac{1}{8})$ be fixed. Using a standard localization procedure, we express $\boldsymbol{\omega}^+$ as a sum $\sum_{m,n\in\mathbb{Z}}\boldsymbol{\omega}^{mn}$, where each function $\boldsymbol{\omega}^{mn}$ is supported in $K^{mn}_{2\epsilon} \times \mathbb{R}$, the sum equals $\boldsymbol{\omega}^+$ in $K^{mn}_{-2\epsilon} \times \mathbb{R}$, and we have the structure $\boldsymbol{\omega}^{mn} = \eta^{mn}\boldsymbol{\omega}^+ - \mathbf{V}^{mn}$ where η^{mn} are infinitely differentiable cut-off functions (the partition of unity) and $\mathbf{V}^{mn} = (V^{mn}_1, V^{mn}_2, 0)$ are corrections which ensure that $\operatorname{div}\boldsymbol{\omega}^{mn} = 0$. The term $\|A^{1/2}\boldsymbol{\omega}^+\|^2_{2;\mathbb{R}^3}$ can now be written in this form:

$$\big\|A^{1/2}\boldsymbol{\omega}^+\big\|^2_{2;\mathbb{R}^3} = \big(A\boldsymbol{\omega}^+, \boldsymbol{\omega}^+\big)_{2;\mathbb{R}^3} = \big(\mathbf{curl}\,\boldsymbol{\omega}^+, \boldsymbol{\omega}^+\big)_{2;\mathbb{R}^3} = \sum_{m,n\in\mathbb{Z}}\sum_{k,l\in\mathbb{Z}}\big(\mathbf{curl}\,\boldsymbol{\omega}^{mn}, \boldsymbol{\omega}^{kl}\big)_{2;\mathbb{R}^3}. \tag{7}$$

The last sum is in fact only the sum over $k \in \{m - 1; m; m + 1\}$ and $l \in \{n - 1; n; n + 1\}$ because otherwise the supports of $\boldsymbol{\omega}^{mn}$ and $\boldsymbol{\omega}^{kl}$ have empty intersections.

We denote by $(-\Delta)_{mn}$ the operator $-\Delta$ with the domain $D((-\Delta)_{mn}) := W^{2,2}(C^{mn}) \cap W^{1,2}_0(C^{mn})$. Further, we denote by y^{kl}_{mn} the solution of the 2D Neumann problem

$$\Delta_{2D}\,y^{kl}_{mn} = -(-\Delta)^{1/4}_{mn}\big(\partial_3\omega^{kl}_3\big) \quad \text{in } K^{mn}_2, \qquad \frac{\partial y^{kl}_{mn}}{\partial\mathbf{n}} = 0 \quad \text{on } \partial K^{mn}_2 \tag{8}$$

for each fixed $x_3 \in \mathbb{R}$, $m, n \in \mathbb{Z}$, $k \in \{m-1; m; m+1\}$ and $l \in \{n-1; n; n+1\}$. Function y_{mn}^{kl} satisfies the estimate $\|\nabla_{2D} y_{mn}^{kl}\|_{2;C^{mn}} \leqslant c\|(-\Delta)_{mn}^{1/4} \partial_3 \omega_3^{kl}\|_{2;C^{mn}}$ with constant c independent of m, n, k and l.

For each $x_3 \in \mathbb{R}$, we define an auxiliary function z_{mn}^{kl} to be the solution of the equation

$$\nabla_{2D}^{\perp} z_{mn}^{kl} = (-\Delta)_{mn}^{1/4} \boldsymbol{\omega}_{2D}^{kl} - \nabla_{2D} y_{mn}^{kl} \tag{9}$$

in K_2^{mn}. The solution exists because $\nabla_{2D} \cdot [(-\Delta)_{mn}^{1/4} \boldsymbol{\omega}_{2D}^{kl} - \nabla_{2D} y_{mn}^{kl}] = 0$. It can be proven that z_{mn}^{kl} is constant on ∂K_2^{mn}. Moreover, z_{mn}^{kl} is unique up to an additive function of t and x_3. We can now choose this function so that $z_{mn}^{kl} = 0$ on ∂K_2^{mn}. Applying also the estimate satisfied by function y_{mn}^{kl}, one can deduce that $\|z_{mn}^{kl}\|_{2;C^{mn}} \leqslant c\|(-\Delta)_{mn}^{1/4} \boldsymbol{\omega}^{kl}\|_{2;C^{mn}} + c\|(-\Delta)_{mn}^{1/4}(\partial_3 \omega_3^{kl})\|_{2;C^{mn}}$.

Due to the definition of y_{mn}^{kl} and z_{mn}^{kl}, the vector function $(-\Delta)_{mn}^{1/4} \boldsymbol{\omega}^{kl}$ has the form

$$(-\Delta)_{mn}^{1/4} \boldsymbol{\omega}^{kl} = \big(\partial_1 y_{mn}^{kl}, \partial_2 y_{mn}^{kl}, (-\Delta)_{mn}^{1/4} \omega_3^{kl}\big) + \mathbf{curl}\big(0, 0, z_{mn}^{kl}\big) \quad \text{in } C^{mn}.$$

Thus, we can rewrite the term $(\mathbf{curl}\, \boldsymbol{\omega}^{mn}, \boldsymbol{\omega}^{kl})_{2;C^{mn}}$ on the right hand side of (7) as follows:

$$\begin{aligned} \big(\mathbf{curl}\, \boldsymbol{\omega}^{mn}, \boldsymbol{\omega}^{kl}\big)_{2;C^{mn}} &= \int_{C^{mn}} (-\Delta)_{mn}^{-1/4}\, \mathbf{curl}\, \boldsymbol{\omega}^{mn} \cdot (-\Delta)_{mn}^{1/4} \boldsymbol{\omega}^{kl}\, \mathrm{d}\mathbf{x} \\ &= \int_{C^{mn}} \big[(-\Delta)^{-1/4}\, \mathbf{curl}\, \boldsymbol{\omega}^{mn} \cdot \big(\partial_1 y_{mn}^{kl}, \partial_2 y_{mn}^{kl}, (-\Delta)_{mn}^{1/4} \omega_3^{kl}\big) + (-\Delta)^{3/4} \boldsymbol{\omega}^{mn} \cdot \big(0, 0, z_{mn}^{kl}\big)\big]\, \mathrm{d}\mathbf{x}. \end{aligned} \tag{10}$$

Since $\|\nabla_{2D} y_{mn}^{kl}\|_{2;C^{mn}}$ and $\|z_{mn}^{kl}\|_{2;C^{mn}}$ are estimated from above by $\|(-\Delta)_{mn}^{1/4}(\partial_3 \omega_3^{kl})\|_{2;C^{mn}}$, we observe that the right hand side of (10) is controlled by certain norms of ω_3^{mn} or ω_3^{kl}, multiplied by the norm of $\boldsymbol{\omega}^{mn}$. Using the structure of $\boldsymbol{\omega}^{mn}$, we finally obtain

$$\big(\mathbf{curl}\, \boldsymbol{\omega}^{mn}, \boldsymbol{\omega}^{kl}\big)_{2;C^{mn}} \leqslant \delta c \big\|\boldsymbol{\omega}^+\big\|_{1/2,2;C^{mn}}^2 + c(\delta) \big\|\omega_3^+\big\|_{3/2,2;C^{mn}}^2. \tag{11}$$

The sum over $m, n \in \mathbb{Z}$ on the right hand side of (7) can be split to altogether 25 parts: the parts successively correspond to $m, n \in \mathbb{Z}$; $m = 0 \mod 5, \ldots, m = 4 \mod 5$ and $n = 0 \mod 5, \ldots, n = 4 \mod 5$. Indices k, l always run over $k \in \{m-1; m; m+1\}$, $l \in \{n-1; n; n+1\}$. Let us denote the sum over m, n such that $m = 0 \mod 5$ and $n = 0 \mod 5$ by $\sum_{m,n}^{(1)}$. The corresponding domains C^{mn} are disjoint and their union equals $\mathbb{R}^3$ up to the set of measure zero. Applying (11), we get

$$\begin{aligned} \sum_{m,n\in\mathbb{Z}}^{(1)} \sum_{k,l} \big(\mathbf{curl}\, \boldsymbol{\omega}^{mn}, \boldsymbol{\omega}^{kl}\big)_{2;C^{mn}} &\leqslant \delta c \sum_{m,n\in\mathbb{Z}}^{(1)} \big\|\boldsymbol{\omega}^+\big\|_{1/2,2;C^{mn}}^2 + c(\delta) \sum_{m,n\in\mathbb{Z}}^{(1)} \big\|\omega_3^+\big\|_{3/2,2;C^{mn}}^2 \\ &\leqslant \delta c \big\|\boldsymbol{\omega}^+\big\|_{1/2,2;\mathbb{R}^3}^2 + c(\delta) \big\|\omega_3^+\big\|_{3/2,2;\mathbb{R}^3}^2. \end{aligned} \tag{12}$$

Estimating the sums over all other $m, n \in \mathbb{Z}$ in the same way and using (7), we derive that $\|A^{1/2} \boldsymbol{\omega}^+\|_{2;\mathbb{R}^3}^2$ is less than or equal to

$$\delta c \big\|\boldsymbol{\omega}^+\big\|_{1/2,2;\mathbb{R}^3}^2 + c(\delta) \big\|\omega_3^+\big\|_{3/2,2;\mathbb{R}^3}^2 \leqslant \delta c \big(\big\|\boldsymbol{\omega}^+\big\|_{2;\mathbb{R}^3}^2 + \big\|A^{1/2} \boldsymbol{\omega}^+\big\|_{2;\mathbb{R}^3}^2\big) + c(\delta) \big\|\omega_3^+\big\|_{3/2,2;\mathbb{R}^3}^2.$$

Choosing $\delta > 0$ so small that $\delta c \leqslant \frac{1}{2}$, and using condition (ii), we deduce that $\|A^{1/2} \boldsymbol{\omega}^+\|_{2;\mathbb{R}^3}^2$ is integrable in the interval $(0, T)$. This information, together with (6), enables us to complete the proof of Theorem 1.

The proof of Theorem 2 can be performed similarly. If condition (i)′ holds then we only need to express $\|A^{1/2} \boldsymbol{\omega}^+\|_{2;\mathbb{R}^3}^2 \equiv \int_0^\infty \lambda\, \mathrm{d}(F_\lambda \boldsymbol{\omega}^+, \boldsymbol{\omega}^+)_{2;\mathbb{R}^3} = \int_0^a \ldots + \int_a^\infty \ldots$ (recall that F_λ is the resolution of identity associated with A) and to treat the integrals on the right hand side separately. The case of condition (ii)′ requires a more detailed analysis.

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