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A constructive method for the stabilization of the wave equation with localized Kelvin–Voigt damping

Une méthode constructive pour la stabilisation de l'équation des ondes avec un amortissement localisé de type Kelvin–Voigt

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ABSTRACT

We consider the wave equation with Kelvin–Voigt damping in a bounded domain. The damping is localized in a suitable open subset of the domain under consideration. The exponential stability result proposed by Liu and Rao for that system assumes that the damping is localized in a neighborhood of the whole boundary, and the damping coefficient is continuously differentiable with a bounded Laplacian. We propose a new solution to the exponential stability problem based on the introduction of a new variable, and a constructive frequency domain approach. The main features of our method are: (i) the damping region need not be a neighborhood of the whole boundary; (ii) the damping coefficient is assumed to be bounded measurable with bounded measurable gradient only; (iii) the introduction of a new variable. These features enable us to improve on the damping coefficient smoothness and more especially on the feedback control region. Further, when combined with a recent result of Borichev and Tomilov on the polynomial decay of bounded semigroups, the new method enables us to prove a polynomial decay estimate of the energy when the damping coefficient is bounded measurable only.

RÉSUMÉ

On considère l'équation des ondes avec un amortissement de type Kelvin–Voigt dans un domaine borné. L'amortissement est localisé dans un sous-ensemble ouvert convenablement choisi dans le domaine en question. Le résultat de stabilité exponentielle proposé par Liu et Rao pour ce système suppose que l'amortissement est localisé dans un voisinage de tout le bord, et le coefficient de l'amortissement est continûment dérivable avec un laplacien borné. Nous proposons une solution nouvelle au problème de la stabilité exponentielle basée sur l'introduction d'une nouvelle variable et une méthode constructive de type domaine des fréquences. Les caractéristiques principales de notre approche sont : (i) il n'est pas nécessaire que la région où l'amortissement est localisé soit un voisinage de tout le bord ; (ii) le coefficient d'amortissement ainsi que son gradient sont supposés seulement bornés et mesurables ; (iii) l'introduction d'une nouvelle variable. Ces éléments nous permettent d'améliorer la régularité du coefficient d'amortissement, et plus particulièrement la région du contrôle dissipatif. De plus, si on combine la nouvelle méthode avec un résultat récent de Borichev et Tomilov sur la décroissance polynômiale de semi-groupes,

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cela nous permet de démontrer une estimation de décroissance polynômiale de l'énergie lorsque le coefficient d'amortissement est seulement borné et mesurable.

Version française abrégée

On considère l'équation des ondes localement amortie (2) pour laquelle on démontre les résultats de stabilisation suivants :

Théorème 1. *Soit ω un ouvert non vide dans Ω. On suppose que le coefficient d'amortissement a est une fonction bornée, mesurable, et strictement positive sur ω. Alors, l'opérateur $\mathcal{A}$ associé a (2) génère un semi-groupe continu de contractions $(S(t)_{t\geqslant 0})$ fortement stable sur l'espace de Hilbert $\mathcal{H}$ (voir (7)).*

De plus, si a vérifie (1), et ω satisfait la contrainte géométrique (GC) ci-dessous, alors on a l'estimation de décroissance polynômiale (8).

Théorème 2. *Supposons que ω satisfait la contrainte géométrique (GC) ci-dessous. Quant au coefficient d'amortissement a, on suppose, en plus de (1), que $a \in W^{1,\infty}(\Omega)$ avec $|\nabla a(x)|^2 \leqslant M_0 a(x)$, pour presque tout x dans Ω, pour une certaine constante strictement positive M_0. Alors, le semi-groupe $(S(t)_{t\geqslant 0})$ vérifie l'estimation de décroissance exponentielle (9).*

1. Introduction and statement of the main results

The stabilization of the wave equation with a locally distributed dissipation (the damping is localized in a suitable nonvoid subset of the domain under consideration) has been the subject of extensive research ever since the pioneering work of [18] on this topic. As can easily be checked in the literature, most of the problems discussed in this framework deal with viscous damping of the form ay_t or $ag(y_t)$, e.g. [3–6,9,10,12,16,18–22]. When it comes to the case of locally distributed viscoelastic damping of Kelvin–Voigt type, the literature is poorly documented; only a few papers discuss this particular topic, e.g. [13–15]. What makes this problem interesting is the fact that the operator defining the damping is unbounded, and the solution of the corresponding dynamical system does not have the requisite smoothness to apply the usual multiplier method, which makes the corresponding stabilization problem more difficult to study, especially in the multidimensional setting. Actually, to the author's knowledge, there are only two papers in the literature concerning the multidimensional problem, one of the papers being a note announcing the full article [14,15].

The purpose of this note is to contribute to a solution of that problem. For the sequel, we need some notations. Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geqslant 1)$ having a sufficiently smooth boundary $\Gamma = \partial\Omega$. Let $a \in L^\infty(\Omega)$, be a nonnegative function satisfying:

$$\exists a_0 > 0\colon a(x) \geqslant a_0, \quad \text{a.e. } x \in \omega, \tag{1}$$

where ω is an open set contained in Ω. The subset ω will be made more precise in the sequel. Throughout the note, we denote by $|u|_r$ the norm of a function $u \in L^r(\Omega)$, $1 \leqslant r \leqslant \infty$.

Consider the following damped wave equation

$$\begin{cases} y_{tt} - \Delta y - div(a\nabla y_t) = 0 & \text{in } \Omega \times (0,\infty), \\ y = 0 \quad \text{on } \Gamma \times (0,\infty), \qquad y(0) = y^0, & y_t(0) = y^1 \quad \text{in } \Omega. \end{cases} \tag{2}$$

System (2) corresponds to the wave equation with viscoelastic damping when $a \equiv 1$, and in this case, the underlying semigroup is known to be analytic. However system (2) is subject to condition (1) which restricts the effectiveness of the damping term $-div(a\nabla y_t)$ to the set where $a > 0$; this model may be viewed as a model of interaction between an elastic material (portion of Ω where $a \equiv 0$), and a viscoelastic material (portion of Ω where $a > 0$).

Let $\{y^0, y^1\} \in H_0^1(\Omega) \times L^2(\Omega)$. System (2) is then well-posed in the space $H_0^1(\Omega) \times L^2(\Omega)$ [15].

Introduce the energy

$$E(t) = \frac{1}{2}\int_\Omega \left\{ \left|y_t(x,t)\right|^2 + \left|\nabla y(x,t)\right|^2 \right\} dx, \quad \forall t \geqslant 0. \tag{3}$$

The energy E is a nonincreasing function of the time variable t as we have for almost every $t \geqslant 0$,

$$E'(t) = -\int_\Omega a(x)\left|\nabla y_t(x,t)\right|^2 dx. \tag{4}$$

Setting $Z = \binom{y}{y'}$, (2) may be recast as: $Z' - \mathcal{A}Z = 0$ in $(0, \infty)$, $Z(0) = \binom{y^0}{y^1}$, where the unbounded operator $\mathcal{A}$ is given by

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \Delta & div(a\nabla .) \end{pmatrix} \tag{5}$$

with $D(\mathcal{A}) = \{(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega);\ \Delta u + div(a\nabla v) \in L^2(\Omega)\}$.

Introduce the Hilbert space over the field $\mathbb{C}$ of complex numbers $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$, equipped with the norm

$$\|Z\|_{\mathcal{H}}^2 = \int_\Omega \left\{|v|^2 + |\nabla u|^2\right\} dx, \quad \forall Z = (u, v) \in \mathcal{H}. \tag{6}$$

We now introduce a geometric constraint (GC) on the subset ω where the dissipation is effective; we proceed as in Liu [12] (see also [8,11]).

(GC). There exist open sets $\Omega_j \subset \Omega$ with piecewise smooth boundary $\partial\Omega_j$, and points $x_0^j \in \mathbb{R}^N$, $j = 1, 2, \ldots, J$, such that $\Omega_i \cap \Omega_j = \emptyset$, for any $1 \leqslant i < j \leqslant J$, and:

$$\Omega \cap \mathcal{N}_\delta\left[\left(\bigcup_{j=1}^J \Gamma_j\right) \cup \left(\Omega \setminus \bigcup_{j=1}^J \Omega_j\right)\right] \subset \omega,$$

for some $\delta > 0$, where $\mathcal{N}_\delta(S) = \bigcup_{x\in S}\{y \in \mathbb{R}^N;\ |x - y| < \delta\}$, for $S \subset \mathbb{R}^N$, $\Gamma_j = \{x \in \partial\Omega_j;\ (x - x_0^j) \cdot \nu^j(x) > 0\}$, ν^j being the unit normal vector pointing into the exterior of Ω_j.

Our main results read:

Theorem 1.1. *Suppose that ω is an arbitrary nonempty open set in Ω. Let the damping coefficient a be bounded measurable, and positive in ω. The operator $\mathcal{A}$ generates a C_0 semigroup of contractions $(S(t)_{t\geqslant 0})$ on $\mathcal{H}$, which is strongly stable*:

$$\lim_{t\to\infty} \left\|S(t)Z^0\right\|_{\mathcal{H}} = 0, \quad \forall Z^0 \in \mathcal{H}. \tag{7}$$

Furthermore, if a satisfies (1), *and ω satisfies the geometric constraint* (*GC*) *above, then we have the polynomial decay estimate*:

$$\exists C_0 > 0\colon \left\|S(t)Z^0\right\|_{\mathcal{H}} \leqslant \frac{C_0\|Z^0\|_{D(\mathcal{A})}}{(1+t)^{\frac{1}{2}}}, \quad \forall t \geqslant 0,\ \forall Z^0 \in D(\mathcal{A}). \tag{8}$$

Theorem 1.2. *Suppose that ω satisfies the geometric constraint* (*GC*) *above. As for the damping coefficient a, further assume that $a \in W^{1,\infty}(\Omega)$ with $|\nabla a(x)|^2 \leqslant M_0 a(x)$, for almost every x in Ω, for some positive constant M_0. The semigroup $(S(t)_{t\geqslant 0})$ is exponentially stable,* viz.*, there exist positive constants M and λ with*

$$\left\|S(t)Z^0\right\|_{\mathcal{H}} \leqslant M \exp(-\lambda t)\left\|Z^0\right\|_{\mathcal{H}}, \quad \forall Z^0 \in \mathcal{H}. \tag{9}$$

Remark 1.3. The second part of Theorem 1.1 on the polynomial decay of the energy is in sharp contrast with what happens in the case of a viscous damping of the form ay_t; in fact, when (GC) holds, the geometric control condition of Bardos, Lebeau and Rauch [1] is met, and exponential decay of the energy should be expected. However, it was shown in the one-dimensional setting that exponential decay of the energy fails if the coefficient a is discontinuous along the interface [13]; one would expect the same conclusion to hold in the multidimensional setting, but this is not done yet.

The earlier result in the literature establishing the exponential decay of the energy in the multidimensional setting assumes that ω is a neighborhood of the whole boundary, and the damping coefficient a satisfies [14,15]: $a \in C^{1,1}(\bar{\Omega})$, $\Delta a \in L^\infty(\Omega)$, and $|\nabla a(x)|^2 \leqslant M_0 a(x)$, for almost every x in Ω, for some positive constant M_0. It can be checked that if (u, v) lies in $D(\mathcal{A})$, then u lies in $H_0^1(\Omega)$ only, as opposed to $u \in H^2(\Omega) \cap H_0^1(\Omega)$ in the case of frictional damping ay_t; this lost of derivative makes it very difficult to apply the usual multiplier technique. Before doing so, a change of variable is needed; the change of variable in [15] led its authors to impose more constraints on the damping coefficient and especially on the feedback control region than necessary. In this regard, Theorem 1.2 strongly improves the exponential stability result of [14,15].

2. Ideas for proving Theorems 1.1 and 1.2

The proof of the first part of Theorem 1.1 about the semigroup generation, and its strong stability may be found in [15]. Additionally it is also shown in [15] that the imaginary axis is contained in the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$. Thanks to a recent result [2, Theorem 2.4], the proof of the second part then amounts to showing the resolvent estimate

$\|(ib\mathcal{I}-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}=O(|b|^2)$ as $|b|\nearrow+\infty$. To this end, let $U\in\mathcal{H}$, and let b be a real number with $|b|\geqslant 1$. Since the range of $ib\mathcal{I}-\mathcal{A}$ is $\mathcal{H}$, there exists $Z\in D(\mathcal{A})$ such that

$$ibZ-\mathcal{A}Z=U. \tag{10}$$

We will sketch the proof of the fact that

$$\|Z\|_{\mathcal{H}}\leqslant K_0|b|^2\|U\|_{\mathcal{H}}, \tag{11}$$

where here and in the sequel, K_0 is a generic positive constant that may eventually depend on Ω, ω, and a, but not on b.

To establish (11), first, we note that if $Z=(u,v)$, and $U=(f,g)$, then (10) may be recast as

$$\begin{aligned} &ibu-v=f,\\ &ibv-\Delta u-div(a\nabla v)=g. \end{aligned}\tag{12}$$

Taking the inner product with Z on both sides of (10), then taking the real parts, we immediately get

$$|\sqrt{a}\nabla v|_2^2\leqslant\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}}. \tag{13}$$

It now follows from the first equation in (12), and (13):

$$b^2|\sqrt{a}\nabla u|_2^2\leqslant 2|\sqrt{a}\nabla v|_2^2+2|\sqrt{a}\nabla f|_2^2\leqslant 2|\sqrt{a}\nabla v|_2^2+K_0\|f\|^2_{H_0^1(\Omega)}\leqslant 2\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}}+K_0\|U\|^2_{\mathcal{H}}. \tag{14}$$

Introduce $u_1=u-w$, where $w=G(div(a\nabla v))$, with $G\in\mathcal{L}(H^{-1}(\Omega);H_0^1(\Omega))$ being the inverse of the negative Laplacian with Dirichlet boundary conditions. Since (u,v) lies in $D(\mathcal{A})$, one easily checks that $u_1\in H^2(\Omega)\cap H_0^1(\Omega)$. Thanks to (13), we note that

$$\|w\|^2_{H_0^1(\Omega)}\leqslant|a|_\infty\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}},\qquad \|u_1\|_{H_0^1(\Omega)}\leqslant\|Z\|_{\mathcal{H}}+\sqrt{|a|_\infty\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}}}. \tag{15}$$

On the other hand, the second equation in (12) becomes

$$ibv-\Delta u_1=g. \tag{16}$$

It immediately follows from (16):

$$|b|\|v\|_{H^{-1}(\Omega)}\leqslant\|u_1\|_{H_0^1(\Omega)}+K_0|g|_2. \tag{17}$$

Now for each $j=1,\dots,J$, where J appears in the geometric constraint (GC) stated above, set $m^j(x)=x-x_0^j$ and $R_j=\sup\{|m^j(x)|,x\in\Omega\}$. Let $0<\delta_0<\delta_1<\delta$, where δ is the one given in (GC). Set:

$$S=\left(\bigcup_{j=1}^J\Gamma_j\right)\cup\left(\Omega\setminus\bigcup_{j=1}^J\Omega_j\right),\qquad Q_0=\mathcal{N}_{\delta_0}(S),\qquad Q_1=\mathcal{N}_{\delta_1}(S),\qquad \omega_1=\Omega\cap Q_1,$$

and for each j, let φ_j be a function satisfying $\varphi_j\in W^{1,\infty}(\Omega)$, $0\leqslant\varphi_j\leqslant 1$, $\varphi_j=1$ in $\bar{\Omega}_j\setminus Q_1$, $\varphi_j=0$ in $\Omega\cap Q_0$.

Let $\alpha>0$ and β be real constants with $\alpha(N-2)<\beta<\alpha N$. Multiply (16) by $\beta\bar{u}_1$, integrate on Ω, and take real parts to get

$$\beta\Re\int_\Omega g\bar{u}_1\,dx=\beta\Re\int_\Omega(ibv-\Delta u_1)\bar{u}_1\,\mathrm{d}x=\beta\|u_1\|^2_{H_0^1(\Omega)}+\beta\Re\int_\Omega v(ib\bar{u}-ib\bar{w})\,\mathrm{d}x. \tag{18}$$

Using (12), it follows: $\beta\Re\int_\Omega v(ib\bar{u}-ib\bar{w})\,\mathrm{d}x=\beta\Re\int_\Omega v(-\bar{v}-\bar{f}-ib\bar{w})\,\mathrm{d}x$. Hence

$$\beta\Re\int_\Omega g\bar{u}_1\,\mathrm{d}x=\beta\|u_1\|^2_{H_0^1(\Omega)}-\beta|v|_2^2-\beta\Re\int_\Omega v(\bar{f}+ib\bar{w})\,\mathrm{d}x. \tag{19}$$

It follows from (15) and (17): $|\beta\Re\int_\Omega v(\bar{f}+ib\bar{w})\,\mathrm{d}x|\leqslant K_0(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}}+\|U\|_{\mathcal{H}}^{\frac12}\|Z\|_{\mathcal{H}}^{\frac32}+\|U\|_{\mathcal{H}}^{\frac32}\|Z\|_{\mathcal{H}}^{\frac12})$. Whence

$$K_0\big(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}}+\|U\|_{\mathcal{H}}^{\frac12}\|Z\|_{\mathcal{H}}^{\frac32}+\|U\|_{\mathcal{H}}^{\frac32}\|Z\|_{\mathcal{H}}^{\frac12}\big)\geqslant\beta\|u_1\|^2_{H_0^1(\Omega)}-\beta|v|_2^2. \tag{20}$$

Now, multiply (16) by $2\alpha\varphi_j m^j\cdot\nabla\bar{u}_1$, integrate on Ω_j, and take real parts to get

$$2\alpha\Re\int_{\Omega_j}g\big(\varphi_j m^j\cdot\nabla\bar{u}_1\big)\,\mathrm{d}x=2\alpha\Re\int_{\Omega_j}v\varphi_j m^j\cdot\nabla(-\bar{v}-\bar{f}-ib\bar{w})\,\mathrm{d}x-2\alpha\Re\int_{\Omega_j}\Delta u_1\big(\varphi_j m^j\cdot\nabla\bar{u}_1\big)\,\mathrm{d}x. \tag{21}$$

The application of Green's formula shows

$$-2\alpha\Re\int_{\Omega_j} v\varphi_j m^j\cdot\nabla\bar v\,\mathrm{d}x = \alpha N\int_{\Omega_j}\varphi_j|v|^2\,\mathrm{d}x + \alpha\int_{\Omega_j}\big(m^j\cdot\nabla\varphi_j\big)|v|^2 - 2\alpha\int_{\partial\Omega_j}\varphi_j\big(m^j\cdot\nu^j\big)|v|^2\,\mathrm{d}\Gamma, \tag{22}$$

and

$$\begin{aligned}-2\alpha\Re\int_{\Omega_j}\Delta u_1\big(\varphi_j m^j\cdot\nabla\bar u_1\big)\,\mathrm{d}x &= 2\alpha\Re\int_{\Omega_j}(\nabla u_1\cdot\nabla\varphi_j)m^j\cdot\nabla\bar u_1\,\mathrm{d}x - (N-2)\alpha\int_{\Omega_j}\varphi_j|\nabla u_1|^2\,\mathrm{d}x\\ &\quad - \alpha\int_{\Omega_j}\big(m^j\cdot\nabla\varphi_j\big)|\nabla u_1|^2\,\mathrm{d}x + \alpha\int_{\partial\Omega_j}\varphi_j\big(m^j\cdot\nu^j\big)|\nabla u_1|^2\,\mathrm{d}\Gamma\\ &\quad - 2\alpha\Re\int_{\partial\Omega_j}\left(\frac{\partial u_1}{\partial\nu^j}\right)\varphi_j m^j\cdot\nabla\bar u_1\,\mathrm{d}\Gamma.\end{aligned}\tag{23}$$

It can be checked that the boundary integral in (22) vanishes while

$$\alpha\int_{\partial\Omega_j}\varphi_j\big(m^j\cdot\nu^j\big)|\nabla u_1|^2\,\mathrm{d}\Gamma - 2\alpha\Re\int_{\partial\Omega_j}\left(\frac{\partial u_1}{\partial\nu^j}\right)\varphi_j m^j\cdot\nabla\bar u_1\,\mathrm{d}\Gamma \geqslant 0.$$

Hence, taking the sum over j, one derives from (21)–(23), and (1):

$$\begin{aligned}K_0\big(\|U\|_{\mathcal H}\|Z\|_{\mathcal H} + \|U\|_{\mathcal H}^{\frac32}\|Z\|_{\mathcal H}^{\frac12}\big) &\geqslant \alpha N\int_\Omega |v|^2\,\mathrm{d}x - (N-2)\alpha\|u_1\|^2_{H_0^1(\Omega)} - K_0\int_\Omega a|\nabla v|^2\,\mathrm{d}x\\ &\quad - K_0\int_\Omega a|\nabla u_1|^2\,\mathrm{d}x - 2\alpha\Re ib\sum_{j=1}^J\int_{\Omega_j} v\varphi_j m^j\cdot\nabla\bar w\,\mathrm{d}x.\end{aligned}\tag{24}$$

The estimates established so far are conducted the same way in the proofs of Theorems 1.1 and 1.2. Only estimating the utmost right term in the lower line of (24) requires different approaches. In the case of Theorem 1.1, one easily derives, thanks to (15):

$$\left|2\alpha\Re ib\sum_{j=1}^J\int_{\Omega_j} v\varphi_j m^j\cdot\nabla\bar w\,\mathrm{d}x\right| \leqslant K_0|b|\|U\|_{\mathcal H}^{\frac12}\|Z\|_{\mathcal H}^{\frac32}. \tag{25}$$

Combining (1), (14), (20), (24), (25), and using Young inequality, one derives (11). Applying [2, Theorem 2.4], one gets the claimed polynomial decay estimate, thereby completing the proof sketch of Theorem 1.1. As for Theorem 1.2, estimating that term is more intricate; this is where the smoothness assumption on the damping coefficient a is needed. Before proceeding with that estimate, we note that we shall now show that for U, Z as above, and all b, one has

$$\|Z\|_{\mathcal H} \leqslant K_0\|U\|_{\mathcal H}, \tag{26}$$

from which will follow the claimed exponential decay estimate thanks to, e.g. [7, Theorem 3], or [17, Corollary 4]. To this end, and as noted above, it remains to estimate the utmost right term in the lower line of (24). For each j, introduce $z_j\in H_0^1(\Omega)$ with $-\Delta z_j = div(\varphi_j m^j v)$ in Ω. One checks that for each j

$$\Re ib\int_{\Omega_j} v\varphi_j m^j\cdot\nabla\bar w\,\mathrm{d}x = -\Re ib\int_{\Omega_j} av\varphi_j m^j\cdot\nabla\bar v\,\mathrm{d}x - \Re ib\int_\Omega(\nabla a\cdot\nabla z_j)\bar v\,\mathrm{d}x, \tag{27}$$

from which one derives

$$\left|2\alpha\Re ib\sum_{j=1}^J\int_{\Omega_j} v\varphi_j m^j\cdot\nabla\bar w\,\mathrm{d}x\right| \leqslant K_0|b||\sqrt a v|_2\big(\|U\|_{\mathcal H}^{\frac12}\|Z\|_{\mathcal H}^{\frac12} + \|Z\|_{\mathcal H}\big). \tag{28}$$

Now, multiply the second equation in (12) by $-iba\bar v$, apply Green's theorem over Ω, use the first equation in (12), and take real parts to derive $b^2|\sqrt a v|_2^2 = \Re\int_\Omega\{\nabla(v+f)\cdot\nabla(a\bar v) + iba(\nabla v\cdot\nabla a)\bar v - ibga\bar v\}\,\mathrm{d}x$. Applying Hölder inequality, then Young inequality, we finally get

$$b^2|\sqrt a v|_2^2 \leqslant K_0\big(\|U\|_{\mathcal H}\|Z\|_{\mathcal H} + \|U\|_{\mathcal H}^{\frac32}\|Z\|_{\mathcal H}^{\frac12} + \|U\|_{\mathcal H}^{\frac12}\|Z\|_{\mathcal H}^{\frac32} + \|U\|_{\mathcal H}^2\big). \tag{29}$$

Reporting (29) in (28), we find

$$\left|2\alpha\Re ib\sum_{j=1}^{J}\int_{\Omega_j} v\varphi_j m^j\cdot\nabla\bar{w}\,\mathrm{d}x\right|\leqslant K_0\big(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}}+\|U\|_{\mathcal{H}}^{\frac{3}{2}}\|Z\|_{\mathcal{H}}^{\frac{1}{2}}+\|U\|_{\mathcal{H}}^{\frac{5}{4}}\|Z\|_{\mathcal{H}}^{\frac{3}{4}}$$
$$+\|U\|_{\mathcal{H}}^{\frac{3}{4}}\|Z\|_{\mathcal{H}}^{\frac{5}{4}}+\|U\|_{\mathcal{H}}^{\frac{1}{2}}\|Z\|_{\mathcal{H}}^{\frac{3}{2}}+\|U\|_{\mathcal{H}}^{\frac{1}{4}}\|Z\|_{\mathcal{H}}^{\frac{7}{4}}\big). \tag{30}$$

The combination of (1), (14), (20), (24), (30), and the application of Young inequality lead to (26) for all b with $|b|\geqslant 1$. Next, using the continuity of the resolvent operator yields (26) for all b, thereby completing our proof sketch. □

References

[1] C. Bardos, G. Lebeau, J. Rauch, Sharp sufficient conditions for the observation, control and stabilization from the boundary, SIAM J. Control Optim. 30 (1992) 1024–1065.

[2] A. Borichev, Y. Tomilov, Optimal polynomial decay of functions and operator semigroups, Math. Ann. 347 (2010) 455–478.

[3] G. Chen, S.A. Fulling, F.J. Narcowich, S. Sun, Exponential decay of energy of evolution equations with locally distributed damping, SIAM J. Appl. Math. 51 (1991) 266–301.

[4] C.M. Dafermos, Asymptotic behaviour of solutions of evolution equations, in: M.G. Crandall (Ed.), Nonlinear Evolution Equations, Academic Press, New York, 1978, pp. 103–123.

[5] X. Fu, Longtime behavior of the hyperbolic equations with an arbitrary internal damping, Z. Angew. Math. Phys. 62 (2011) 667–680.

[6] A. Haraux, Une remarque sur la stabilisation de certains systèmes du deuxième ordre en temps, Port. Math. 46 (1989) 245–258.

[7] F.L. Huang, Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces, Ann. Differential Equations 1 (1985) 43–56.

[8] V. Komornik, Exact Controllability and Stabilization. The Multiplier Method, RAM, Masson & John Wiley, Paris, 1994.

[9] I. Lasiecka, D. Toundykov, Energy decay rates for the semilinear wave equation with nonlinear localized damping and source terms, Nonlinear Anal. 64 (2006) 1757–1797.

[10] G. Lebeau, Equation des ondes amorties, in: Algebraic and Geometric Methods in Mathematical Physics, Kaciveli, 1993, in: Math. Phys. Stud., vol. 19, Kluwer Acad. Publ., Dordrecht, 1996, pp. 73–109.

[11] J.L. Lions, Contrôlabilité exacte, perturbations et stabilisation des systèmes distribués, vol. 2, RMA, Masson, Paris, 1988.

[12] K. Liu, Locally distributed control and damping for the conservative systems, SIAM J. Control Optim. 35 (1997) 1574–1590.

[13] K. Liu, Z. Liu, Exponential decay of energy of the Euler–Bernoulli beam with locally distributed Kelvin–Voigt damping, SIAM J. Control Optim. 36 (3) (1998) 1086–1098.

[14] K. Liu, B. Rao, Stabilité exponentielle des équations des ondes avec amortissement local de Kelvin–Voigt, C. R. Acad. Sci. Paris, Ser. I 339 (2004) 769–774.

[15] K. Liu, B. Rao, Exponential stability for the wave equations with local Kelvin–Voigt damping, Z. Angew. Math. Phys. 57 (2006) 419–432.

[16] M. Nakao, Decay of solutions of the wave equation with a local nonlinear dissipation, Math. Ann. 305 (1996) 403–417.

[17] J. Prüss, On the spectrum of C_0-semigroups, Trans. Amer. Math. Soc. 284 (1984) 847–857.

[18] J. Rauch, M. Taylor, Exponential decay of solutions to hyperbolic equations in bounded domains, Indiana Univ. Math. J. 24 (1974) 79–86.

[19] L.R. Tcheugoué Tébou, Stabilization of the wave equation with localized nonlinear damping, J. Differential Equations 145 (1998) 502–524.

[20] L.R. Tcheugoué Tébou, Well-posedness and energy decay estimates for the damped wave equation with L^r localizing coefficient, Comm. Partial Differential Equations 23 (1998) 1839–1855.

[21] L. Tebou, A Carleman estimates based method for the stabilization of some locally damped semilinear hyperbolic equations, ESAIM Control Optim. Calc. Var. 14 (2008) 561–574.

[22] E. Zuazua, Exponential decay for the semilinear wave equation with locally distributed damping, Comm. Partial Differential Equations 15 (1990) 205–235.